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THE BRAUER-PICARD GROUP OF THE REPRESENTATION CATEGORY OF FINITE SUPERGROUP ALGEBRAS

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Abstract. We develop further the techniques presented in a previous article (M. Mombelli. *Abh. Math. Semin. Univ. Hamb.* 82 (2012), 173–192), to study bimodule categories over the representation categories of arbitrary finite-dimensional Hopf algebras. We compute the Brauer-Picard group of equivalence classes of exact invertible bimodule categories over the representation categories of a certain large family of pointed non-semisimple Hopf algebras, the so called *supergroup algebras* (N. Andruskiewitsch, P. Etingof and S. Gelaki. *Michigan Math. J.* 49 (2001), 277–298). To obtain this result we first give a classification of equivalence classes of exact indecomposable bimodule categories over such Hopf algebras.

1. Introduction

The Brauer-Picard group $\mathrm{BrPic}(\mathcal{C})$ of a finite tensor category $\mathcal{C}$, introduced in [9], is the group of equivalence classes of invertible exact $\mathcal{C}$-bimodule categories. This group is a fundamental piece of information needed to compute extensions of a given tensor category by a finite group. Also it has a close relation to certain structures appearing in mathematical physics, see for example [7, 14].

In [9] the authors compute the Brauer-Picard group of the representation category of an arbitrary finite Abelian group G. Given two semisimple bimodule categories $\mathcal{M}, \mathcal{N}$ over $\mathrm{Rep}(G)$ the authors compute the decomposition into indecomposable bimodule categories of the tensor product $\mathcal{M} \boxtimes_{\mathrm{Rep}(G)} \mathcal{N}$. Using this result and some other techniques they compute $\mathrm{BrPic}(\mathrm{Rep}(G))$. The same methods appear to be unsuccessful for an arbitrary finite-dimensional Hopf algebra H. The problem of explicitly given a decomposition of the tensor product $\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} \mathcal{N}$ into indecomposable bimodule categories for arbitrary bimodule categories $\mathcal{M}$, $\mathcal{N}$ looks complicated.

Using Hopf theoretic techniques this problem was partially solved in [17] by considering the tensor product $\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} \mathcal{N}$ only in the case when both bimodule categories $\mathcal{M}$, $\mathcal{N}$ are invertible.

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The main result of this paper is the computation of the Brauer-Picard group of the representation category of the so called *supergroup algebras* over Abelian groups.

Let G be a finite group, u be an element of order 2 in the center of G and V be a finite-dimensional G-module such that u acts by -1 in V. The vector space V is a Yetter-Drinfeld module over G by declaring the coaction $\delta : V \to \Bbbk G \otimes_{\Bbbk} V$, $\delta(v) = u \otimes v$, $v \in V$. The Nichols algebra of V is the exterior algebra $\wedge(V)$ and the bosonization $\wedge(V)\#\Bbbk G$ is called a *supergroup algebra* [1]. We shall denote this Hopf algebra by $\mathcal{A}(V, u, G)$. This family of Hopf algebras played a central role in the classification of finite-dimensional triangular Hopf algebras [8].

If H is a finite-dimensional Hopf algebra then left module categories over $\mathrm{Rep}(H)$ are parametrized by equivalence classes of certain H-comodule algebras. Since bimodule categories over $\mathrm{Rep}(H)$ are the same as left module categories over the Deligne's tensor [5] product $\mathrm{Rep}(H) \boxtimes \mathrm{Rep}(H)^{\mathrm{op}} = \mathrm{Rep}(H \otimes_{\Bbbk} H^{\mathrm{cop}})$, then bimodule categories over $\mathrm{Rep}(H)$ are parametrized by equivalence classes of certain left $H \otimes_{\Bbbk} H^{\mathrm{cop}}$-comodule algebras. If $\mathcal{M}$ and $\mathcal{N}$ are invertible exact $\mathrm{Rep}(H)$-bimodule categories the tensor product $\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} \mathcal{N}$ is an invertible exact $\mathrm{Rep}(H)$-bimodule category, therefore indecomposable. In Section 4 we collect all these results and we recall results from [17] allowing us to give a precise description of the category $\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} \mathcal{N}$.

If H is a coradically graded Hopf algebra then $H \otimes_{\Bbbk} H^{\mathrm{cop}}$ is also coradically graded, and indecomposable exact left module categories over $\mathrm{Rep}(H \otimes_{\Bbbk} H^{\mathrm{cop}})$ are parametrized by certain equivalence classes of deformations of coideal subalgebras in $H \otimes_{\Bbbk} H^{\mathrm{cop}}$. This results are contained in Section 5.

If $\mathcal{M}$ is an exact indecomposable bimodule category over $\mathrm{Rep}(\mathcal{A}(V, u, G))$ then there exists a certain left $\mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G)^{\mathrm{cop}}$-comodule algebra K such that $\mathcal{M}$ is equivalent to the category of finite-dimensional left K-modules. Since $\mathcal{A}(V, u, G)$ is a coradically graded Hopf algebra then K is a certain deformation of a coideal subalgebra of $\mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G)^{\mathrm{cop}}$. In Section 6 we explicitly describe coideal subalgebras in the tensor product $\mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G)^{\mathrm{cop}}$. Using these results, in Section 7, we prove that if $\mathcal{M}$ is an exact indecomposable left module category over the category $\mathrm{Rep}(\mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G)^{\mathrm{cop}})$ there exists a 6-tuple $(W^1, W^2, W^3, \beta, F, \psi)$ where

(i) $F \subseteq G \times G$ is a subgroup, $\psi \in Z^2(F, \Bbbk^\times)$ is a 2-cocycle,
(ii) $W^1, W^2 \subseteq V$ $W^3 \subseteq V \oplus V$ are subspaces such that $W^3 \cap W^1 \oplus W^2 = 0$, $W^3 \cap V \oplus 0 = 0 = W^3 \cap 0 \oplus V$, and all subspaces are invariant under the action of F,
(iii) $\beta : \oplus_{i=1}^3 W^i \times \oplus_{i=1}^3 W^i \to \Bbbk$ is a bilinear form stable under the action of F, such that

$$\begin{aligned}
\beta(w_1, w_2) &= -\beta(w_2, w_1),\\
\beta(w_1, w_3) &= \beta(w_3, w_1),\\
\beta(w_2, w_3) &= -\beta(w_3, w_2),
\end{aligned}$$

for all $w_i \in W^i$, $i = 1, 2, 3$, and β restricted to $W^i \times W^i$ is symmetric for any $i = 1, 2, 3$. If $(u, u) \notin F$ then β restricted to $W^1 \times W^2$ is null,

such that $\mathcal{M}$ is module equivalent to the category of finite-dimensional left $\mathcal{K}(W^1, W^2, W^3, \beta, F, \psi)$-modules, where $\mathcal{K}(W^1, W^2, W^3, \beta, F, \psi)$ is a certain left comodule algebra over $\mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G)^{\mathrm{cop}}$. We also describe equivalence classes of such module categories.

Using these results, in Section 8, we prove our main result:

Theorem 1.1. *Assume G is Abelian. The group* $\mathrm{BrPic}(\mathrm{Rep}(\mathcal{A}(V, u, G)))$ *is isomorphic to the group of (certain equivalence classes of) pairs (T, α) where*

- $\alpha \in O(G \oplus \widehat{G})$*, see Definition 8.1,*
- $T : V \oplus V^* \to V \oplus V^*$ *is a linear isomorphism such that*

$$T(v, f) = x^{-1} \cdot T(y \cdot v, y \cdot f),$$

$$T^1(0, f) = 0, \quad T^2(0, f)(T^1(v, 0)) = f(v),$$

for all $(v, f) \in V \oplus V^*$, $(x, y) \in U_\alpha$. Here $T(v, f) = (T^1(v, f), T^2(v, f))$ for all $f \in V^*, v \in V$.

The product of two such pairs (T, α), (T', α') is

$$(T, \alpha) \bullet (T', \alpha') = (T \circ T', \alpha\alpha').$$

As expected, this group is not finite, as is the case for fusion categories. The main difficulty to prove this theorem relies on finding which of the comodule algebras $\mathcal{K}(W^1, W^2, W^3, \beta, F, \psi)$ give invertible bimodule categories and give an explicit description of the product of the group $\mathrm{BrPic}(\mathrm{Rep}(\mathcal{A}(V, u, G)))$. Most of Section 8 is dedicated to this task.

It is expected that this result led us to construct interesting new families of finite non-semisimple tensor categories that are extensions by a finite group of the category $\mathrm{Rep}(\mathcal{A}(V, u, G))$.

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2. Notation and preliminaries

We shall work over an algebraically closed field $\Bbbk$ of characteristic 0. All vector spaces and algebras are considered over $\Bbbk$. We denote $vect_{\Bbbk}$ the category of finite-dimensional $\Bbbk$-vector spaces. If A is an algebra we shall denote by ${}_A\mathcal{M}$ ($\mathcal{M}_A$) the category of finite-dimensional left (right) A-modules.

If V is a vector space any bilinear form $\beta : V \times V \to \Bbbk$ determines a linear morphism $\widehat{\beta} : V \to V^*$

$$\widehat{\beta}(v)(w) = \beta(v, w), \quad \text{for all } v, w \in V. \tag{2.1}$$

Let $\mathcal{M}$ be an Abelian category. A full subcategory $\mathcal{N}$ of $\mathcal{M}$ is called a *Serre subcategory* if

- every object in $\mathcal{M}$ isomorphic to an object in $\mathcal{N}$ is in $\mathcal{N}$,
- every $\mathcal{M}$-quotient and every $\mathcal{M}$-subobject of an object in $\mathcal{N}$ lies in $\mathcal{N}$,
- every $\mathcal{M}$-extension of objects in $\mathcal{N}$ lies in $\mathcal{N}$.

It is well-known that if $F : \mathcal{M} \to \mathcal{M}$ is an exact functor then the full subcategory of objects $N \in \mathcal{M}$ such that $F(N) = 0$ is a Serre subcategory. This fact will be used without further mention.

2.1. **Finite tensor categories.** A *tensor category over* $\Bbbk$ is a $\Bbbk$-linear Abelian rigid monoidal category. Hereafter all tensor categories will be assumed to be over a field $\Bbbk$. A *finite category* is an Abelian $\Bbbk$-linear category such that it has only a finite number of isomorphism classes of simple objects, Hom spaces are finite-dimensional $\Bbbk$-vector spaces, all objects have finite lenght and every simple object has a projective cover. A *finite tensor category* [10] is a tensor category with finite underlying Abelian category such that the unit object is simple. All functors will be assumed to be $\Bbbk$-linear and all categories will be finite.

2.2. **Twisting comodule algebras.** Let H be a Hopf algebra. Let us recall that a Hopf 2-cocycle for H is a map $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$, invertible with respect to convolution, such that

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}), \tag{2.2}$$

$$\sigma(x, 1) = \varepsilon(x) = \sigma(1, x), \tag{2.3}$$

for all $x, y, z \in H$. Using this cocycle there is a new Hopf algebra structure constructed over the same coalgebra H with the product described by

$$x \cdot_{[\sigma]} y = \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)})\ x_{(2)}y_{(2)}, \qquad x, y \in H. \tag{2.4}$$

This new Hopf algebra is denoted by $H^{[\sigma]}$. If $\sigma : H \otimes H \to \Bbbk$ is a Hopf 2-cocycle and A is a left H-comodule algebra, then we can define a new product in A by

$$a ._\sigma b = \sigma(a_{(-1)}, b_{(-1)})\, a_{(0)}.b_{(0)}, \tag{2.5}$$

$a, b \in A$. We shall denote by A_σ this new algebra with unchanged left H-comodule structure.

Lemma 2.1. *The algebra A_σ is a left $H^{[\sigma]}$-comodule algebra.* □

3. Representations of finite tensor categories

Let $\mathcal{C}$ be a tensor category. For the definition and basic notions of left and right exact module categories we refer to [10, 19].

In this paper we only consider module categories that are finite categories. A module functor between left $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{M}'$ over a tensor category

$\mathcal{C}$ is a pair (T, c), where $T : \mathcal{M} \to \mathcal{M}'$ is a functor and $c_{X,M} : T(X\overline{\otimes}M) \to X\overline{\otimes}T(M)$ is a family of natural isomorphism such that for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$:

$$(\mathrm{id}_X \otimes c_{Y,M})c_{X,Y\overline{\otimes}M}T(m_{X,Y,M}) = m_{X,Y,T(M)}\, c_{X\otimes Y,M} \tag{3.1}$$

$$\ell_{T(M)}\, c_{\mathbf{1},M} = T(\ell_M). \tag{3.2}$$

The direct sum of two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ is the $\Bbbk$-linear category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinate-wise module structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories. Any exact module category is equivalent to a direct sum of indecomposable exact module categories, see [10].

Definition 3.1. [2, 11] Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. A *submodule category* of $\mathcal{M}$ is a Serre subcategory stable under the action of $\mathcal{C}$.

The next Lemma is a straightforward consequence of the definitions.

Lemma 3.2.

1. *Let $\mathcal{M}$ be an exact $\mathcal{C}$-module category and $\mathcal{N} \subseteq \mathcal{M}$ a submodule category. If $\mathcal{M} = \oplus_{i\in I}\mathcal{M}_i$ is a decomposition into indecomposable module categories then there is a subset $J \subseteq I$ such that $\mathcal{N} = \oplus_{i\in J}\mathcal{M}_i$.*
2. *If $\mathcal{M}$ is an indecomposable exact $\mathcal{C}$-module category and $(F, c) : \mathcal{N} \to \mathcal{M}$ is a $\mathcal{C}$-module functor such that F is full and faithful, and the subcategory $F(\mathcal{N})$ is Serre then F is an equivalence.* □

3.1. **Bimodule categories.** Let $\mathcal{C}, \mathcal{D}$ be tensor categories. For the definition of a $(\mathcal{C}, \mathcal{D})$-*bimodule category* we refer to [13, 9]. A $(\mathcal{C}, \mathcal{D})$-bimodule category is the same as left $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$-module category. Here $\boxtimes$ denotes Deligne's tensor product of Abelian categories [5].

A $(\mathcal{C}, \mathcal{D})$-bimodule category is *decomposable* if it is the direct sum of two non-trivial $(\mathcal{C}, \mathcal{D})$-bimodule categories. A $(\mathcal{C}, \mathcal{D})$-bimodule category is *indecomposable* if it is not decomposable. A $(\mathcal{C}, \mathcal{D})$-bimodule category is *exact* if it is exact as a left $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$-module category.

If $\mathcal{M}$ is a right $\mathcal{C}$-module category then $\mathcal{M}^{\mathrm{op}}$ denotes the opposite Abelian category with left $\mathcal{C}$ action $\mathcal{C} \times \mathcal{M}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$, $(M, X) \mapsto M\overline{\otimes}X^*$ and associativity isomorphisms $m^{\mathrm{op}}_{X,Y,M} = m^{-1}_{Y^*,X^*,M}$ for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. Similarly if $\mathcal{M}$ is a left $\mathcal{C}$-module category. If $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category then $\mathcal{M}^{\mathrm{op}}$ is a $(\mathcal{D}, \mathcal{C})$-bimodule category. See [13, Prop. 2.15].

If $\mathcal{M}, \mathcal{N}$ are $(\mathcal{C}, \mathcal{D})$-bimodule categories, a *bimodule functor* is the same as a module functor of $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$-module categories, that is a functor $F : \mathcal{M} \to \mathcal{N}$ such that $(F, c) : \mathcal{M} \to \mathcal{N}$ is a functor of left $\mathcal{C}$-module categories, also $(F, d) : \mathcal{M} \to \mathcal{N}$ is a functor of right $\mathcal{D}$-module categories and

$$(\mathrm{id}_X \otimes d_{M,Y})c_{X,M\overline{\otimes}_r Y}F(\gamma_{X,M,Y}) = \gamma_{X,F(M),Y}(c_{X,M} \otimes \mathrm{id}_Y)d_{X\overline{\otimes}_l M,Y}, \tag{3.3}$$

for all $M \in \mathcal{M}$, $X \in \mathcal{C}$, $Y \in \mathcal{D}$.

A $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$ is called *invertible* [9, Prop. 4.2] if there are equivalences of bimodule categories

$$\mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}, \quad \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\mathrm{op}} \simeq \mathcal{C}.$$

Lemma 3.3. [9, Corollary 4.4] *If $\mathcal{M}$ is an invertible $(\mathcal{C},\mathcal{D})$-bimodule category then it is indecomposable as a bimodule category.* □

Lemma 3.4. [9, Prop. 4.2] *Let $\mathcal{M}$ be an exact $(\mathcal{C},\mathcal{D})$-bimodule category. The following statements are equivalent.*

1. *$\mathcal{M}$ is an invertible.*
2. *There exists a $\mathcal{D}$-bimodule equivalence $\mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}$.*
3. *There exists a $\mathcal{C}$-bimodule equivalence $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\mathrm{op}} \simeq \mathcal{C}$.*
4. *The functor $R : \mathcal{D}^{\mathrm{op}} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$, $R(X)(M) = M\overline{\otimes}X$, for all $X \in \mathcal{D}$, $M \in \mathcal{M}$, is an equivalence of tensor categories.*
5. *The functor $L : \mathcal{C} \to \mathrm{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{M})$, $L(Y)(M) = Y\overline{\otimes}M$, for all $Y \in \mathcal{C}$, $M \in \mathcal{M}$, is an equivalence of tensor categories.*

Proof. The proof of [9, Prop. 4.2] extends *mutatis mutandis* to the non-semisimple case using results from [10]. □

3.2. **Module categories over Hopf algebras.** Let H be a finite-dimensional Hopf algebra and let $(\mathcal{A},\lambda)$ be a left H-comodule algebra. The category ${}_{\mathcal{A}}\mathcal{M}$ is a representation of $\mathrm{Rep}(H)$. The action

$$\overline{\otimes} : \mathrm{Rep}(H) \times {}_{\mathcal{A}}\mathcal{M} \to {}_{\mathcal{A}}\mathcal{M}, \quad V\overline{\otimes}M = V\otimes_{\Bbbk}M,$$

for all $V \in \mathrm{Rep}(H)$, $M \in {}_{\mathcal{A}}\mathcal{M}$. The left $\mathcal{A}$-module structure on $V\otimes_{\Bbbk}M$ is given by

$$a \cdot (v\otimes m) = a_{(-1)} \cdot v\otimes a_{(0)} \cdot m,$$

for all $a \in \mathcal{A}$, $v \in V$, $m \in M$. Here $\lambda : \mathcal{A} \to H\otimes_{\Bbbk}\mathcal{A}$, $\lambda(a) = a_{(-1)}\otimes a_{(0)}$.

If $\mathcal{A}$ is a H-comodule algebra via $\lambda : \mathcal{A} \to H\otimes_{\Bbbk}\mathcal{A}$, we shall say that a (right) ideal J is H-costable if $\lambda(J) \subseteq H\otimes_{\Bbbk}J$. We shall say that $\mathcal{A}$ is (right) H-simple, if there is no nontrivial (right) ideal H-costable in $\mathcal{A}$. When $\mathcal{A}$ is right H-simple then the category ${}_{\mathcal{A}}\mathcal{M}$ is exact, see [3, Prop. 1.20].

Theorem 3.5. [3, Theorem 3.3] *Let $\mathcal{M}$ be an exact indecomposable module category over* $\mathrm{Rep}(H)$ *then there exists a left H-comodule algebra $\mathcal{A}$ right H-simple with trivial coinvariants such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ as* $\mathrm{Rep}(H)$*-modules.* □

Two H-comodule algebras $\mathcal{A}$, $\mathcal{A}'$ are *equivariantly Morita equivalent* if the module categories ${}_{\mathcal{A}'}\mathcal{M}$, ${}_{\mathcal{A}}\mathcal{M}$ are equivalent.

4. Bimodule categories over Hopf algebras

4.1. **Tensor product of invertible bimodule categories.** Let A, B be finite-dimensional Hopf algebras. A $(\mathrm{Rep}(B), \mathrm{Rep}(A))$-bimodule category is the same as a left $\mathrm{Rep}(B\otimes_{\Bbbk}A^{\mathrm{cop}})$-module category. This follows from the fact that $\mathrm{Rep}(A)^{\mathrm{op}} \simeq \mathrm{Rep}(A^{\mathrm{cop}})$ and $\mathrm{Rep}(B) \boxtimes \mathrm{Rep}(A^{\mathrm{cop}}) \simeq \mathrm{Rep}(B\otimes_{\Bbbk}A^{\mathrm{cop}})$. Thus Theorem 3.5 implies that any exact indecomposable $(\mathrm{Rep}(B), \mathrm{Rep}(A))$-bimodule category is equivalent to the category ${}_{S}\mathcal{M}$ of finite-dimensional left S-modules, where S is a finite-dimensional right $B\otimes_{\Bbbk}A^{\mathrm{cop}}$-simple left $B\otimes_{\Bbbk}A^{\mathrm{cop}}$-comodule algebra.

Since $\mathrm{Rep}(A)$ is canonically a $\mathrm{Rep}(A)$-bimodule category then there exists some right $A\otimes_{\Bbbk}A^{\mathrm{cop}}$-simple left $A\otimes_{\Bbbk}A^{\mathrm{cop}}$-comodule algebra $\mathcal{A}$ such that $\mathrm{Rep}(A) \simeq {}_{\mathcal{A}}\mathcal{M}$

as $\operatorname{Rep}(A)$-bimodule categories. In [17] we computed this comodule algebra. Let us recall this result.

We denote by $\operatorname{diag}(A)$ the left $A \otimes_{\Bbbk} A^{\mathrm{cop}}$-comodule algebra with underlying algebra A and comodule structure:

$$\lambda : \operatorname{diag}(A) \to A \otimes_{\Bbbk} A^{\mathrm{cop}} \otimes_{\Bbbk} \operatorname{diag}(A), \quad \lambda(a) = a_{(1)} \otimes a_{(3)} \otimes a_{(2)},$$

for all $a \in A$. Thus the category ${}_A\mathcal{M}$ is a $\operatorname{Rep}(A)$-bimodule category.

Lemma 4.1.

1. $\operatorname{diag}(A)$ *is a right simple left* $A \otimes_{\Bbbk} A^{\mathrm{cop}}$*-comodule algebra and* $\operatorname{diag}(A)^{\mathrm{co}\, A \otimes_{\Bbbk} A^{\mathrm{cop}}} = \Bbbk 1$.
2. *There is an equivalence of* $\operatorname{Rep}(A)$*-bimodule categories*

$${}_A\mathcal{M} \simeq {}_{\operatorname{diag}(A)}\mathcal{M}.$$

Proof. 1. Let $0 \neq I \subseteq A$ be a right ideal A-costable. Then for any $a \in I$, $a_{(1)} \otimes a_{(3)} \otimes a_{(2)} \in A \otimes_{\Bbbk} A \otimes I$ which implies that $a_{(1)} \otimes a_{(2)} \in A \otimes_{\Bbbk} I$. Thus I is a right ideal stable under the coaction, then $I = A$.

2. The identity functor $\mathrm{Id} : {}_A\mathcal{M} \to {}_{\operatorname{diag}(A)}\mathcal{M}$ is an equivalence of $\operatorname{Rep}(A)$-bimodule categories. □

Let us recall some constructions and results obtained in [17] concerning the tensor product of bimodule categories over Hopf algebras. Set $\pi_A : A \otimes B \to A$, $\pi_B : A \otimes B \to B$ the algebra maps defined by

$$\pi_A(x \otimes y) = \epsilon(y)x, \quad \pi_B(x \otimes y) = \epsilon(x)y,$$

for all $x \in A, y \in B$.

Let K be a right $B \otimes A^{\mathrm{cop}}$-simple left $B \otimes A^{\mathrm{cop}}$-comodule algebra and L a right $A \otimes B^{\mathrm{cop}}$-simple left $A \otimes B^{\mathrm{cop}}$-comodule algebra. Thus the category ${}_K\mathcal{M}$ is a $(\operatorname{Rep}(B), \operatorname{Rep}(A))$-bimodule category and ${}_L\mathcal{M}$ is a $(\operatorname{Rep}(A), \operatorname{Rep}(B))$-bimodule category.

The category $\mathcal{M}(A, B, K, \overline{L})$ is the category ${}^B_K\mathcal{M}_{\overline{L}}$ of $(K, \overline{L})$-bimodules and left B-comodules such that the comodule structure is a bimodule morphism. See [17, Section 3]. It has a structure of $(\operatorname{Rep}(A), \operatorname{Rep}(A))$-bimodule category. Recall that $\overline{L}$ is the left $B \otimes A^{\mathrm{cop}}$-comodule algebra with opposite algebra structure L^{op} and left $B \otimes A^{\mathrm{cop}}$-comodule structure:

$$\overline{\lambda} : L \to A^{\mathrm{cop}} \otimes_{\Bbbk} B \otimes_{\Bbbk} L, \quad l \mapsto (\mathcal{S}_B^{-1} \otimes \mathcal{S}_A)(l_{(-1)}) \otimes l_{(0)}, \tag{4.1}$$

for all $l \in L$. Also L is a right B-comodule with comodule map given by

$$l \mapsto l_{(0)} \otimes \pi_B(l_{(-1)}), \tag{4.2}$$

for all $l \in L$, and K is a left B-comodule with comodule map given by

$$k \mapsto \pi_B(k_{(-1)}) \otimes k_{(0)}, \tag{4.3}$$

for all $k \in K$. Using this structure we can form the cotensor product $L \square_B K$. Define

$$\lambda(l \otimes k) = \pi_A(l_{(-1)}) \otimes \pi_A(k_{(-1)}) \otimes l_{(0)} \otimes k_{(0)}, \tag{4.4}$$

for all $l\otimes k \in L\square_B K$. Then $L\square_B K$ is a left $A\otimes_{\Bbbk} A^{\mathrm{cop}}$-comodule algebra. See [17, Lemma 3.6].

In [17] we have presented the functors

$$\mathcal{F} : {}_{L\square_B K}\mathcal{M} \to \mathcal{M}(A, B, K, \overline{L}), \quad \mathcal{G} : \mathcal{M}(A, B, K, \overline{L}) \to {}_{L\square_B K}\mathcal{M}$$

by $\mathcal{F}(N) = (L\otimes_{\Bbbk} K)\otimes_{L\square_B K} N$ for all $N \in {}_{L\square_B K}\mathcal{M}$ and $\mathcal{G}(M) = M^{\mathrm{co}\, B}$ for all $M \in \mathcal{M}(A, B, K, \overline{L})$. We recall that the left B-comodule structure on $\mathcal{F}(N)$ is given by $\delta : \mathcal{F}(N) \to B\otimes_{\Bbbk}\mathcal{F}(N)$,

$$\delta(l\otimes k\otimes n) = \pi_B(k_{(-1)})\mathcal{S}^{-1}(\pi_B(l_{(-1)}))\otimes l_{(0)}\otimes k_{(0)}\otimes n, \tag{4.5}$$

for all $l \in L, k \in K, n \in N$.

This pair of functors were studied in [6, 4]. In the following theorem we summarize some results from [17].

Theorem 4.2.

(a) *There is a* $\mathrm{Rep}(A)$*-bimodule equivalence:*

$$ {}_L\mathcal{M} \boxtimes_{\mathrm{Rep}(B)} {}_K\mathcal{M} \simeq \mathcal{M}(A, B, K, \overline{L}).$$

(b) $\mathcal{F}$ *and* $\mathcal{G}$ *are* $\mathrm{Rep}(A)$*-bimodule functors.*

(c) *Assume that both bimodule categories* ${}_L\mathcal{M}$, ${}_K\mathcal{M}$ *are invertible and* $L\otimes_{\Bbbk} K \simeq C\otimes_{\Bbbk} L\square_B K$*, as right* $L\square_B K$*-modules and left* B*-comodules. Here* C *is a certain left* B*-comodule. Then there is an equivalence of* $\mathrm{Rep}(A)$*-bimodule categories*

$${}_{L\square_B K}\mathcal{M} \simeq {}_L\mathcal{M} \boxtimes_{\mathrm{Rep}(B)} {}_K\mathcal{M}.$$

Proof. For the proof of (a) and (b) see [17, Corollary 4.4, Prop. 4.7].

(c). We shall prove that the functors $\mathcal{F}$, $\mathcal{G}$ establish an equivalence of module categories.

Let us prove that $\mathcal{F}(\mathcal{G}(M)) \simeq M$ for all $M \in \mathcal{M}(A, B, K, \overline{L})$. For any $M \in \mathcal{M}(A, B, K, \overline{L})$ there is a projection

$$\pi_M : (L\otimes_{\Bbbk} K)\otimes_{L\square_B K} M^{\mathrm{co}\, B} \to M, \quad \pi_M(l\otimes k\otimes m) = (l\otimes k)\cdot m,$$

for all $l\otimes k \in L\otimes_{\Bbbk} K$, $m \in M^{\mathrm{co}\, B}$. Define the functor $\Phi : \mathcal{M}(A, B, K, \overline{L}) \to vect_{\Bbbk}$, $\Phi(M) = \ker(\pi_M)$. The functor Φ is a module functor. To see this it is enough to prove that the diagram

$$\begin{array}{ccc} \mathcal{F}(\mathcal{G}(X\overline{\otimes}M)) & \xrightarrow{\simeq} & X\overline{\otimes}\mathcal{F}(\mathcal{G}(M)) \\ {\scriptstyle \pi_{X\overline{\otimes}M}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_X\otimes\pi_M} \\ X\otimes M & \xrightarrow[\mathrm{id}]{} & X\otimes M. \end{array}$$

is commutative. Then Φ is exact. The full subcategory $\mathcal{N}$ of $\mathcal{M}(A, B, K, \overline{L})$ consisting of objects M such that $\Phi(M) = 0$ is a submodule category. $\mathcal{N}$ is not the null category since $\pi_{L\otimes_{\Bbbk} K} = \mathrm{id}$, thus $L\otimes_{\Bbbk} K \in \mathcal{N}$. Since both ${}_L\mathcal{M}$, ${}_K\mathcal{M}$ are invertible the product ${}_L\mathcal{M} \boxtimes_{\mathrm{Rep}(B)} {}_K\mathcal{M} \simeq \mathcal{M}(A, B, K, \overline{L})$ is indecomposable. Hence $\mathcal{N} = \mathcal{M}(A, B, K, \overline{L})$. This implies that $\mathcal{F}(\mathcal{G}(M)) = M$ for all $M \in \mathcal{M}(A, B, K, \overline{L})$.

Since $L\otimes_{\Bbbk}K \simeq C\otimes_{\Bbbk}L\Box_B K$, as right $L\Box_B K$-modules and left B-comodules the functor $\mathcal{F}$ is full and faithful, thus it is an equivalence of categories. $\square$

Remark 4.3. In all examples the assumption $L\otimes_{\Bbbk}K \simeq C\otimes_{\Bbbk}L\Box_B K$ in Theorem 4.2 (c) seems to be superfluous, although I do not know any counterexample.

5. Graded comodule algebras over Hopf algebras

From the discussion on Section 3.2, equivalence classes of indecomposable exact module categories over the representation categories of Hopf algebras are in correspondence with equivariant Morita equivalence classes of right simple comodule algebras. To study this class of algebras we developed a technique in [16] using the Loewy filtration and the associated graded algebra. We briefly recall all this notions.

If H is a finite-dimensional Hopf algebra then $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra then the associated graded algebra $\operatorname{gr} H$ is a coradically graded Hopf algebra. If (A,λ) is a left H-comodule algebra, the coradical filtration on H induces a filtration on A, given by $A_n = \lambda^{-1}(H_n\otimes_{\Bbbk}A)$, $n = 1,\dots,m$. This filtration is called the *Loewy series* on A.

Recall that if $H = \oplus_{i=0}^m H(i)$ is a coradically graded Hopf algebra, a left H-comodule algebra (A,λ) is a *graded comodule algebra*, if it is graded as an algebra $A = \oplus_{i=0}^m A(i)$ and for each $0 \le n \le m$

$$\lambda(A(n)) \subseteq \bigoplus_{i=0}^{m} H(i)\otimes_{\Bbbk}A(n-i). \tag{5.1}$$

A graded comodule algebra $A = \oplus_{i=0}^m A(i)$ is *Loewy-graded* if the Loewy series is given by $A_n = \oplus_{i=0}^n A(i)$ for any $0 \le n \le m$.

If A is a left H-comodule algebra the associated graded algebra $\operatorname{gr} A$ obtained from the Loewy filtration is a Loewy-graded left $\operatorname{gr} H$-comodule algebra. For more details see [16].

The following result will be needed later.

Lemma 5.1. *Let $H = \oplus_{i=0}^m H(i)$ be a coradically graded Hopf algebra and (A,λ_A) a left H comodule with a grading $A = \oplus_{i=0}^m A(i)$ such that* (5.1) *holds. If $B \subseteq A$ is a subcomodule algebra and we set $B(n) = B \cap A(n)$ then*

$$B = \oplus_{i=0}^m B(i).$$

Proof. Let $b \in B$, then $b = \sum_{i=0}^m b_i$ where $b_i \in A(i)$. Let us prove that $b_i \in B$ for all $i = 0,\dots,m$. Denote $p : H \to H(0)$, $\pi_j : A \to A(j)$ the canonical projections. Observe that for any $j = 0,\dots,m$

$$(p\otimes\pi_j)\lambda(b) = (p\otimes\mathrm{id})\lambda(b_j).$$

Since $(\epsilon\otimes\mathrm{id})(p\otimes\mathrm{id})\lambda(b_j) = b_j$ then $b_j = (\epsilon\otimes\mathrm{id})(p\otimes\pi_j)\lambda(b) \in B$. $\square$

5.1. **Comodule algebras over coradically graded Hopf algebras.** Let G be a finite group and $H = \oplus_{i=0}^{m} H(i)$ be a finite-dimensional coradically graded Hopf algebra where $H(0) = \Bbbk G$ is the coradical.

Let (A, λ) be a left H-comodule algebra right H-simple with trivial coinvariants and with a grading $A = \oplus_{i=0}^{m} A(i)$ making A a Loewy-graded left H-comodule algebra. Since A is right H-simple with trivial coinvariants then $A(0) = \Bbbk_\psi F$ where $F \subseteq G$ is a subgroup and $\psi \in Z^2(F, \Bbbk^\times)$ is a 2-cocycle.

Set $\pi : A \to A(0)$ the canonical projection and $\epsilon : A(0) \to \Bbbk$ the map given by $\epsilon(e_f) = 1$ for all $f \in F$.

Remark 5.2. If ψ is trivial then $\epsilon : A(0) \to \Bbbk$ is an algebra morphism.

Proposition 5.3. *Assume that ψ is trivial and let $\phi : A \to H$ be the map defined by $\phi = (\mathrm{id}_H \otimes \epsilon\pi)\lambda$. Then*

(i) *ϕ is an algebra morphism,*
(ii) *ϕ is a H-comodule map, and*
(iii) *ϕ is injective.*

Proof. (i). It follows since all maps in the definition of ϕ are algebra morphisms.

(ii). $\Delta\phi = \Delta(\mathrm{id}_H \otimes \epsilon\pi)\lambda = (\mathrm{id}_H \otimes \mathrm{id}_H \otimes \epsilon\pi)(\Delta \otimes \mathrm{id}_A)\lambda$. Using the coassociativity of λ we obtain that $\Delta\phi = (\mathrm{id}_H \otimes \phi)\lambda$.

(iii). Let $a \in \ker\phi$. Assume that $a \neq 0$. Write $a = \sum_{n=0}^{t} a^{(n)}$ where $a^{(n)} \in A(n)$ and $t \leq m$. We can assume that $a^{(t)} \neq 0$. Then $\lambda(a^{(n)}) \in \oplus_{i=0}^{n} H(i) \otimes_\Bbbk A(n-i)$. Set $\lambda(a^{(n)}) = \sum_{i=0}^{n} b_{n,i}$ where $b_{n,i} = \sum_k x_k^{n,i} \otimes c_k^{n,i}$ and $x^{n,i} \in H(i)$, $c^{n,i} \in A(n-i)$.

Since $a^{(t)} \neq 0$ then $b_{t,t} \neq 0$. Indeed, if $b_{t,t} = 0$ then $\lambda(a^{(t)}) \in \oplus_{i=0}^{t-1} H(i) \otimes_\Bbbk A$, hence $a^{(t)} \in \oplus_{i=0}^{t-1} A(i) = A_{t-1}$, which is imposible unless $a^{(t)} = 0$.

Also, $\Delta\phi(a) = 0$, then $(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \epsilon\pi)(\mathrm{id}_H \otimes \lambda)\lambda(a) = 0$ which implies that

$$\sum_{n=0}^{t} \sum_{i=0}^{n} \sum_{k} x_k^{n,i} \otimes (\mathrm{id}_H \otimes \epsilon\pi)\lambda(c_k^{n,i}) = 0$$

The element of the above summation that belongs to $H(t) \otimes_\Bbbk H(0) \otimes_\Bbbk A(0)$ must be equal to zero, hence $\sum_k x_k^{t,t} \otimes (\mathrm{id}_H \otimes \epsilon)\lambda(c_k^{t,t}) = 0$. Since we have that

$$\sum_{k} x_k^{t,t} \otimes (\mathrm{id}_H \otimes \epsilon)\lambda(c_k^{t,t}) = b_{t,t}$$

we get that $b_{t,t} = 0$, which is a contradiction. Therefore $a = 0$. □

In other words, Proposition 5.3 implies that if A is a Loewy-graded right H-simple left comodule algebra with trivial coinvariants and $A(0)$ is a Hopf subalgebra of $H(0)$ then A is isomorphic to a left coideal subalgebra of H. The next step is to study what happens if $A(0)$ is not a Hopf subalgebra of $H(0)$.

Let $\widehat{\psi} \in Z^2(G, \Bbbk^\times)$ be a 2-cocycle such that $\widehat{\psi}\mid_{F \times F} = \psi$.

Lemma 5.4. *There exists a Hopf 2-cocycle* $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ *such that for any homogeneous elements* $x, y \in H$

$$\sigma(x,y) = \begin{cases} \widehat{\psi}(x,y), & \text{if } x,y \in H(0); \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof. See [12, Lemma 4.1]. ☐

The following result is a straightforward consequence of Proposition 5.3.

Lemma 5.5. *Let* A *be a Loewy-graded right* H*-simple left comodule algebra with trivial coinvariants and* $A(0) = \Bbbk_{\psi} F$ *where* $F \subseteq G$ *is a subgroup and* $\psi \in Z^2(F, \Bbbk^{\times})$ *is a 2-cocycle. Then, there exists a Hopf 2-cocycle* $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ *such that* A_{σ} *is isomorphic to a homogeneous left coideal subalgebra of* $H^{[\sigma]}$ *as a left* $H^{[\sigma]}$*-comodule algebras.* ☐

Proof. From Lemma 5.4 there exists a Hopf 2-cocycle $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ such that $\sigma(x,y) = \psi^{-1}(x,y)$ for all $x, y \in F$. The comodule algebra A_{σ} is Loewy-graded and $(A_{\sigma})(0) = \Bbbk F$. Thus the Lemma follows from Proposition 5.3. ☐

6. Supergroup algebras and their coideal subalgebras

We shall recall the definition of supergroup algebras [1], their Hopf algebra structure, and we describe the tensor product of two such Hopf algebras. We compute also their homogeneous coideal subalgebras, a key ingredient to compute module categories.

6.1. **Finite supergroup algebras.**

Let G be a finite group, $u \in G$ be a central element of order 2 and V a finite-dimensional G-module such that $u \cdot v = -v$ for all $v \in V$. The space V has a G-comodule structure $\delta : V \to \Bbbk G \otimes_{\Bbbk} V$ given by $\delta(v) = u \otimes v$, for all $v \in V$. This gives V structure of Yetter-Drinfeld module over $\Bbbk G$. The Nichols algebra of V is the exterior algebra $\mathfrak{B}(V) = \wedge(V)$. The Hopf algebra obtained by bosonization $\wedge(V) \# \Bbbk G$ is called in [1] a *finite supergroup algebra*. We will denote this Hopf algebra by $\mathcal{A}(V, u, G)$. Hereafter we shall denote the element $v \# g$ simply by vg, for all $v \in V, g \in G$.

The algebra $\mathcal{A}(V, u, G)$ is generated by elements $v \in V, g \in G$ subject to relations

$$vw + wv = 0, \quad gv = (g \cdot v)g, \qquad \text{for all } v, w \in V,\ g \in G.$$

The coproduct and antipode are determined by

$$\Delta(v) = v \otimes 1 + u \otimes v, \quad \Delta(g) = g \otimes g,$$

$$\mathcal{S}(v) = -uv, \quad \mathcal{S}(g) = g^{-1},$$

for all $v \in V, g \in G$.

Lemma 6.1. *There is a Hopf algebra isomorphism*

$$\mathcal{A}(V, u, G) \simeq \mathcal{A}(V, u, G)^{\mathrm{cop}}.$$

Proof. Let $\phi : \mathcal{A}(V,u,G) \to \mathcal{A}(V,u,G)$ be the algebra map determined by

$$\phi(v) = vu, \quad \phi(g) = g,$$

for all $v \in V, g \in G$. It follows by a direct computation that ϕ is a Hopf algebra isomorphism between $\mathcal{A}(V,u,G)$ and $\mathcal{A}(V,u,G)^{\mathrm{cop}}$. $\square$

6.2. **Tensor product of supergroup algebras.** Let G_1, G_2 be finite groups and $u_i \in G_i$ be central elements of order 2. For $i = 1,2$ let V_i be finite-dimensional G_i-modules, such that u_i acts in V_i as -1. We shall describe the tensor product Hopf algebra $\mathcal{A}(V_1,u_1,G_1)\otimes_{\Bbbk}\mathcal{A}(V_2,u_2,G_2)$. From now on, we shall denote this Hopf algebra by $\mathcal{A}(V_1,V_2,u_1,u_2,G_1,G_2)$. Let us give a presentation by generators and relations of this algebra.

Set $G = G_1 \times G_2$. Both vector spaces V_1, V_2 are G-modules by setting

$$(g,h)\cdot v_1 = g\cdot v_1, \quad (g,h)\cdot v_2 = h\cdot v_2,$$

for all $(g,h) \in G$, $v_i \in V_i$, $i = 1,2$. The algebra $\mathcal{A}(V_1,V_2,u_1,u_2,G_1,G_2)$ is generated by elements V_1, V_2, G subject to relations

$$v_1w_1 + w_1v_1 = 0, \quad v_2w_2 + w_2v_2 = 0, \quad v_1v_2 = v_2v_1,$$

$$gv_1 = (g\cdot v_1)g, \quad gv_2 = (g\cdot v_2)g,$$

for all $g \in G$, $v_i \in V_i$, $i = 1,2$. The Hopf algebra structure is determined by

$$\Delta(v_1) = v_1\otimes 1 + (u_1,1)\otimes v_1, \quad \Delta(v_2) = v_2\otimes 1 + (1,u_2)\otimes v_2,$$

$$\Delta(g_1,g_2) = (g_1,g_2)\otimes(g_1,g_2),$$

for all $(g_1,g_2) \in G$, $v_i \in V_i$, $i = 1,2$.

We shall define a family of Hopf algebras that are cocycle deformations of tensor product of supergroup algebras. Let $(V_1,V_2,u_1,u_2,G_1,G_2)$ be a data as above. Set $V = V_1 \oplus V_2$. Define $\mathcal{H}(V_1,V_2,u_1,u_2,G_1,G_2) = \wedge(V)\otimes_{\Bbbk}\Bbbk G$ with product determined by

$$vw + wv = 0, \quad gv = (g\cdot v)g, \qquad \text{for any } v,w \in V_1\oplus V_2, g \in G,$$

and coproduct determined by

$$\Delta(v_1) = v_1\otimes 1 + (u_1,1)\otimes v_1, \quad \Delta(v_2) = v_2\otimes 1 + (1,u_2)\otimes v_2,$$

for any $v_i \in V_i$, $i = 1,2$.

Proposition 6.2. *Let $H = \mathcal{A}(V_1,V_2,u_1,u_2,G_1,G_2)$ and $\sigma : H\otimes_{\Bbbk}H \to \Bbbk$ a Hopf 2-cocycle coming from a 2-cocycle $\psi \in Z^2(G,\Bbbk^\times)$ as in Lemma 5.4. Denote $\xi = \psi((u_1,1),(1,u_2))\psi((1,u_2),(u_1,1))^{-1}$. Then*

(i) *if $\xi = 1$ we have $H^{[\sigma]} \simeq \mathcal{A}(V_1,V_2,u_1,u_2,G_1,G_2)$*
(ii) *if $\xi = -1$ then $H^{[\sigma]} \simeq \mathcal{H}(V_1,V_2,u_1,u_2,G_1,G_2)$.*

Proof. Let $v \in V_1, w \in V_2$ then

$$(\mathrm{id}\otimes\Delta)\Delta(v) = v\otimes 1\otimes 1 + (u_1,1)\otimes v\otimes 1 + (u_1,1)\otimes(u_1,1)\otimes v,$$

$$(\mathrm{id}\otimes\Delta)\Delta(w) = w\otimes 1\otimes 1 + (1,u_2)\otimes w\otimes 1 + (1,u_2)\otimes(1,u_2)\otimes w.$$

Therefore, using (2.4), it follows that for any $v_1, w_1 \in V_1$, $v_2, w_2 \in V_2$

$$v_1 \cdot_{[\sigma]} w_1 + w_1 \cdot_{[\sigma]} v_1 = 0, \quad v_2 \cdot_{[\sigma]} w_2 + w_2 \cdot_{[\sigma]} v_2 = 0,$$
$$v_1 \cdot_{[\sigma]} w_2 - \xi\, w_2 \cdot_{[\sigma]} v_1 = 0.$$

Also for any $g \in G$, $i = 1, 2$,

$$g \cdot_{[\sigma]} v_1 = \psi(g, (u_1, 1))\, gv_1, \quad v_1 \cdot_{[\sigma]} g = \psi((u_1, 1), g)\, v_1 g,$$
$$g \cdot_{[\sigma]} v_2 = \psi((1, u_2), g)\, gv_2, \quad v_2 \cdot_{[\sigma]} g = \psi((1, u_2), g)\, v_2 g.$$

Hence

$$g \cdot_{[\sigma]} v \cdot_{[\sigma]} g^{-1} = gvg^{-1},$$

for any $v \in V$. From these relations, and since the coproduct remains unchanged, we deduce that if $\xi = 1$ then $H^{[\sigma]} \simeq \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$ and if $\xi = -1$ then $H^{[\sigma]} \simeq \mathcal{H}(V_1, V_2, u_1, u_2, G_1, G_2)$. ☐

6.3. **Homogeneous coideal subalgebras in supergroup algebras.** A *homogeneous left coideal subalgebra* of a coradically graded Hopf algebra $H = \oplus_{i=0}^m H(i)$ is a left coideal subalgebra $K \subseteq H$ together with an algebra grading $K = \oplus_{i=0}^m K(i)$ such that $K(i) \subseteq H(i)$. The main goal of this section is the classification of homogeneous coideal subalgebras in the tensor product of supergroup algebras.

Let $(V_1, V_2, u_1, u_2, G_1, G_2)$ be a data as in Section 6.2. Denote $V = V_1 \oplus V_2$ and $u = (u_1, u_2) \in G = G_1 \times G_2$. Also set $H = \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$ and $\widetilde{H} = \mathcal{H}(V_1, V_2, u_1, u_2, G_1, G_2)$. If $(v_1, v_2) \in V$ we denote

$$[(v_1, v_2)] = v_1 + v_2 u \in H(1).$$

Remark 6.3. For any $(v_1, v_2) \in V$ we have

$$[(v_1, v_2)]^2 = 0, \quad \Delta([(v_1, v_2)]) = v_1 \otimes 1 + v_2 u \otimes u + (u_1, 1) \otimes [(v_1, v_2)]. \tag{6.1}$$

Definition 6.4. A *coideal subalgebra data* is a collection (W^1, W^2, W^3, F), where

- $W^1 \subseteq V_1$ and $W^2 \subseteq V_2$ are subspaces,
- $W^3 \subseteq V$ is a subspace such that $W^3 \cap W^1 \oplus W^2 = 0$, $W^3 \cap V_1 = 0 = W^3 \cap V_2$,
- $F \subseteq G$ is a subgroup that leaves invariant all subspaces W^i, $i = 1, 2, 3$,
- if $W^3 \neq 0$ we require that $u \in F$.

We denote $C(W^1, W^2, W^3, F)$ the subalgebra of H generated by $\Bbbk F$ and elements in $W^1 \oplus W^2$ and $\{[w] : w \in W^3\}$.

Lemma 6.5. *The algebra $C(W^1, W^2, W^3, F)$ is a homogeneous left coideal subalgebra of H.* ☐

Theorem 6.6. *Let $K = \oplus_{i=0}^m K(i) \subseteq H$ be a homogeneous left coideal subalgebra. There exists a coideal subalgebra data (W^1, W^2, W^3, F) such that $K = C(W^1, W^2, W^3, F)$.*

Proof. Since $K(0) \subseteq \Bbbk G$ is a left coideal subalgebra then $K(0) = \Bbbk F$ for some subgroup $F \subseteq G$. If $K(1) = 0$ then $K = \Bbbk F$. Indeed, if $x \in K(2)$ then $\Delta(x) \in H(0) \otimes_{\Bbbk} K(2) \oplus H(2) \otimes K(0)$, hence $x \in H_1$ and since $H_1 \cap H(2) = 0$ then $x = 0$. In a similar way we can prove that $K(n) = 0$ for all n.

Thus we can assume that $K(1) \neq 0$. The vector space $K(1)$ is a $\Bbbk G$-subcomodule of $V \otimes_{\Bbbk} \Bbbk G$ via

$$(\pi \otimes \mathrm{id})\Delta : K(1) \to \Bbbk G \otimes_{\Bbbk} K(1),$$

where $\pi : H \to \Bbbk G$ is the canonical projection. Thus $K(1) = \oplus_{g \in G} K(1)_g$, where $K(1)_g = \{k \in K(1) : (\pi \otimes \mathrm{id})\Delta(k) = g \otimes k\}$, and

$$K(1)_g \subseteq V_1 \otimes_{\Bbbk} \Bbbk\langle (u_1, 1)g \rangle \oplus V_2 \otimes_{\Bbbk} \Bbbk\langle (1, u_2)g \rangle.$$

Therefore we can write

$$K(1)_{(u_1,1)} = W^1 \oplus \widetilde{W}^2 u \oplus U^3,$$
$$K(1)_{(1,u_2)} = W^2 \oplus \widetilde{W}^1 u \oplus \widetilde{U}^3,$$

where W^1 is the intesection of $K(1)_{(u_1,1)}$ with V_1, $\widetilde{W}^2$ is the intesection of $K(1)_{(u_1,1)}$ with $V_2 \otimes_{\Bbbk} \Bbbk\langle u \rangle$ and U^3 is a direct complement, that is, a vector subspace of $V_1 \oplus V_2 \otimes_{\Bbbk} \Bbbk\langle u \rangle$ consisting of elements of the form $[w]$ where $w \in W^3$ and $W^3 \subseteq V_1 \oplus V_2$. Since $U^3 \cap W^1 \oplus \widetilde{W}^2 u = 0$ then $W^3 \cap W^1 \oplus \widetilde{W}^2 = 0$. The same is done for $K(1)_{(1,u_2)}$, that is W^2 is the intersection of $K(1)_{(1,u_2)}$ with V_2, $\widetilde{W}^1 u$ is the intersection of $K(1)_{(1,u_2)}$ with with $V_1 \otimes_{\Bbbk} \Bbbk\langle u \rangle$ and $\widetilde{U}^3$ is a direct complement. The space $\widetilde{U}^3$ consists of elements of the form $[w]$ where $w \in \widetilde{W}^3$ and $\widetilde{W}^3 \subseteq V_1 \oplus V_2$.

Claim 6.1. *If $u \notin F$ then $\widetilde{W}^2 = \widetilde{W}^1 = \widetilde{W}^3 = W^3 = 0$. We have that $\widetilde{W}^2 = W^2$, $\widetilde{W}^1 = W^1$.*

Proof of Claim. Let $0 \neq (v, w) \in W^3$, then $0 \neq [(v, w)] \in U^3$. Since $\Delta([(v, w)]) \in H(0) \otimes_{\Bbbk} K(1) \oplus H(1) \otimes_{\Bbbk} K(0)$, using (6.1), we get that $u \in F$. The same argument works if $\widetilde{W}^3 \neq 0$, $\widetilde{W}^1 \neq 0$ or if $\widetilde{W}^2 \neq 0$.

Let $0 \neq w \in \widetilde{W}^2$, then $wu \in K(1)_{(u_1,1)}$. Since $u \in F$ then $w \in K(1)_{(1,u_2)}$ and the only possibility is that $w \in W^2$. The other inclusion is proven similarly. Thus $\widetilde{W}^2 = W^2$. The equality $\widetilde{W}^1 = W^1$ follows analogously. □

We claim that $K(1) = W^1 F \oplus W^2 F \oplus U^3 F$. Indeed, take $g \in G$ and $0 \neq w \in K(1)_g$, then

$$w = w_1(u_1, 1)g + w_2(1, u_2)g,$$

for some $w_1 \in V_1, w_2 \in V_2$. Note that

$$\Delta(w) = w_1(u_1, 1)g \otimes (u_1, 1)g + g \otimes w + w_2(1, u_2)g \otimes (1, u_2)g. \tag{6.2}$$

If $w_1 \neq 0$, since $\Delta(w) \in H(0) \otimes_{\Bbbk} K(1) \oplus H(1) \otimes_{\Bbbk} K(0)$, then $(u_1, 1)g \in F$ and $wg^{-1}(u_1, 1) \in K(1)_{(u_1,1)}$. Thus $w \in W^1 F \oplus W^2 F \oplus U^3 F$. If $w_1 = 0$ then $w_2 \neq 0$ and using a same argument as before we conclude that $(1, u_2)g \in F$, thus $wg^{-1}(1, u_2) \in K(1)_{(1,u_2)}$. If $\widetilde{U}^3 = \widetilde{W}^1 = 0$ then $wg^{-1}(1, u_2) \in W^2$ and $w \in W^2 F$. If some of the vector spaces $\widetilde{U}^3$, $\widetilde{W}^1$ are not null then $u \in F$, from which we deduce that $g^{-1}(u_1, 1) \in F$ and $wg^{-1}(u_1, 1) \in K(1)_{(u_1,1)}$. Hence $w \in W^1 F \oplus W^2 F \oplus U^3 F$.

If $S = \{b_i\}$ is any basis of V then H is generated as an algebra by the set

$$\{[b_i], g : b_i \in B, g \in G\}.$$

Indeed, take $v \in V_1$, $w \in V_2$ then $(v,0) = \sum_i \alpha_i\, b_i$, $(0,w) = \sum_i \beta_i\, b_i$ for some families of scalars $\alpha_i, \beta_i \in \Bbbk$, then $v = \sum_i \alpha_i\,[b_i]$ and $w = \sum_i \beta_i\,[b_i]u$. Let $\{b_i : i = 1,\dots,r\}$ be a basis of $W = W^1 \oplus W^2 \oplus W^3$ and extend it to a basis $\{b_i : i = 1,\dots,t\}$, $r \leq t$, of V. Let $n > 1$ and $k \in K(n)$. Write

$$k = \sum_{s_j \in \{0,1\}, g_i \in G} \alpha_{s_1,\dots,s_t,i}\,[b_1]^{s_1}[b_2]^{s_2}\dots[b_t]^{s_t} g_i,$$

for some $\alpha_{s_1,\dots,s_t,i} \in \Bbbk$. Let $p : H \to H(1)$ be the canonical projection. Then $(\mathrm{id}\otimes p)\Delta(k) \in H(n-1)\otimes_{\Bbbk} K(1)$. It follows from a straightforward computation that $(\mathrm{id}\otimes p)\Delta(k)$ is equal to

$$\sum_l \sum_{s_j \in \{0,1\}, g_i \in G} \alpha_{s_1,\dots,s_t,i}\big(\, h_{s_1,\dots,s_t,i}\otimes[b_l]g_i + \widetilde{h}_{s_1,\dots,s_t,i}\otimes[b_l]ug_i\big),$$

for some $0 \neq h_{s_1,\dots,s_t,i}, \widetilde{h}_{s_1,\dots,s_t,i} \in H(n-1)$. This implies that if $r < l$ and $s_l = 1$ then $\alpha_{s_1,\dots,s_t,i} = 0$. Thus K is generated as an algebra by $K(0)$ and $K(1)$, whence $K = C(W^1, W^2, W^3, F)$. ☐

Definition 6.7. If (W^1, W^2, W^3, F) is a coideal subalgebra data denote $\widetilde{C}(W^1, W^2, W^3, F)$ the subalgebra of $\widetilde{H}$ generated by $\Bbbk F$ and elements in $W^1 \oplus W^2$ and $\{[w] : w \in W^3\}$.

Theorem 6.8. *Let $K = \oplus_{i=0}^m K(i) \subseteq \widetilde{H}$ be a homogeneous left coideal subalgebra. There exists a coideal subalgebra data (W^1, W^2, W^3, F) such that $K = \widetilde{C}(W^1, W^2, W^3, F)$.*

Proof. The proof follows the same argument as in the proof of Theorem 6.6. ☐

7. Module categories over tensor product of supergroup algebras

We shall use the same notation as in the previous section, so we have a data $(V_1, V_2, u_1, u_2, G_1, G_2)$ as in subsection 6.2, $H = \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$ and $\widetilde{H} = \mathcal{H}(V_1, V_2, u_1, u_2, G_1, G_2)$. Denote $G = G_1 \times G_2$, $H_i = \mathcal{A}(V_i, u_i, G_i)$ and $u = (u_1, u_2) \in G$.

We shall define a family of comodule algebras over H that will parameterize exact module categories over $\mathrm{Rep}(H)$.

Definition 7.1. We say that the collection (W, β, F, ψ) is a *compatible data* with $(V_1, V_2, u_1, u_2, G_1, G_2)$ if

(i) $W = W^1 \oplus W^2 \oplus W^3$ is a subspace of V such that (W^1, W^2, W^3, F) is a coideal subalgebra data;

(ii) $\beta : W \times W \to \Bbbk$ is a bilinear form stable under the action of F, such that

$$\beta(w_1, w_2) = -\beta(w_2, w_1), \quad \beta(w_1, w_3) = \beta(w_3, w_1),$$
$$\beta(w_2, w_3) = -\beta(w_3, w_2),$$

for all $w_i \in W^i$, $i = 1,2,3$, and β restricted to $W^i \times W^i$ is symmetric for any $i = 1,2,3$;

(iii) if $u \notin F$ then β restricted to $W^1 \times W^2$ and $W^2 \times W^3$ is null;

(iv) $\psi \in Z^2(F, \Bbbk^\times)$.

Given a compatible data (W, β, F, ψ) define $\mathcal{K}(W, \beta, F, \psi)$ as the algebra generated by W and $\{e_f : f \in F\}$, subject to relations

$$e_f e_h = \psi(f,h)\, e_{fh}, \quad e_f w = (f \cdot w) e_f,$$

$$w_i w_j + w_i w_j = \beta(w_i, w_j)1, \quad w_i \in W^i, w_j \in W^j,$$

for any $(i,j) \in \{(1,1),(2,2),(1,3),(3,3)\}$, and relations

$$w_2 w_3 - w_3 w_2 = \beta(w_2, w_3)\, e_u, \quad \text{for any } w_2 \in W^2, w_3 \in W^3,$$

$$w_1 w_2 - w_2 w_1 = \beta(w_1, w_2)\, e_u, \quad \text{for any } w_1 \in W^1, w_2 \in W^2.$$

Define $\lambda : \mathcal{K}(W, \beta, F, \psi) \to H \otimes_\Bbbk \mathcal{K}(W, \beta, F, \psi)$ on the generators

$$\lambda(e_f) = f \otimes e_f, \quad \lambda(w_1) = w_1 \otimes 1 + (u_1, 1) \otimes w_1, \quad \text{for all } f \in F, w_1 \in W^1,$$

$$\lambda(w_2) = w_2 \otimes 1 + (1, u_2) \otimes w_2, \quad \text{for all } w_2 \in W^2,$$

$$\lambda(v,w) = v \otimes 1 + w(1,u_2) \otimes e_u + (u_1, 1) \otimes (v,w), \quad \text{for all } (v,w) \in W^3.$$

Remark 7.2. If (W, β, F, ψ) is a compatible data then W comes with a distinguished decomposition $W = W^1 \oplus W^2 \oplus W^3$. To be more precise one should denote the algebras $\mathcal{K}(W, \beta, F, \psi)$ by $\mathcal{K}(W^1, W^2, W^3, \beta, F, \psi)$. We shall do this only in case we want to emphasize the direct decomposition of W.

Definition 7.3. If $(0, 0, W, F)$ is a coideal subalgebra data and (W, β, F, ψ) is a compatible data with $(V_1, V_2, u_1, u_2, G_1, G_2)$, we shall denote

$$\mathcal{L}(W, \beta, F, \psi) = \mathcal{K}(0, 0, W, \beta, F, \psi).$$

The algebras $\mathcal{L}(W, \beta, F, \psi)$ will be the relevant ones when computing the Brauer-Picard group.

Proposition 7.4. *If (W, β, F, ψ) is a compatible data then $\mathcal{K}(W, \beta, F, \psi)$ is a right H-simple left H-comodule algebra with trivial coinvariants. Also* $\operatorname{gr} \mathcal{K}(W, \beta, F, \psi) = \mathcal{K}(W, 0, F, \psi)$.

Proof. The proof that these algebras are comodule algebras is straightforward. Also, it follows from a direct computation that

$$\operatorname{gr} \mathcal{K}(W, \beta, F, \psi) = \mathcal{K}(W, 0, F, \psi),$$

and $\mathcal{K}(W, \beta, F, \psi)_0 = \Bbbk_\psi F$. Thus, the fact that these algebras are right $\mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$-simple follows from [16, Prop. 4.4]. □

Recall that in Section 4.1 we have defined a left $H_i \otimes_\Bbbk H_i^{\mathrm{cop}}$-comodule algebra $\operatorname{diag}(H_i)$. It follows from Lemma 6.1 that there is an isomorphism of Hopf algebras

$$H_i \otimes_\Bbbk H_i^{\mathrm{cop}} \simeq H_i \otimes_\Bbbk H_i \simeq \mathcal{A}(V_i, V_i, u_i, u_i, G_i, G_i).$$

For any $i = 1, 2$ we shall denote $B_i = \mathcal{A}(V_i, V_i, u_i, u_i, G_i, G_i)$. Also

$$\operatorname{diag}(V_i) = \{(v,v) \in V_i \oplus V_i : v \in V_i\},$$

$$\operatorname{diag}(G_i) = \{(g,g) \in G_i \times G_i : g \in G_i\}.$$

Lemma 7.5. *For any $i = 1, 2$ there is an isomorphism of left B_i-comodule algebras*

$$\operatorname{diag}(H_i) \simeq \mathcal{K}(0 \oplus 0 \oplus \operatorname{diag}(V_i), 0, \operatorname{diag}(G_i), 1).$$

Proof. Define $\sigma : \operatorname{diag}(H_i) \to \mathcal{K}(0 \oplus 0 \oplus \operatorname{diag}(V_i), 0, \operatorname{diag}(G_i), 1)$ as follows. For all $v \in V_i$, $g \in G_i$,

$$\sigma(v) = (v, v)(u_i, u_i), \quad \sigma(g) = (g, g).$$

This gives a well-defined algebra isomorphism. It follows straightforwardly that σ is a B_i-comodule map. □

Remark 7.6. In Lemma 7.5 we write the space $W = 0 \oplus 0 \oplus \operatorname{diag}(V_i)$ to emphasize that $W^1 = 0, W^2 = 0$ and $W^3 = \operatorname{diag}(V_i)$.

Proposition 7.7. *Let $(W, 0, F, \psi)$ be a compatible data and $\widehat{\psi} \in Z^2(G, \Bbbk^\times)$ be a 2-cocycle such that $\widehat{\psi}\mid_F = \psi$. Let $\sigma : H \otimes_\Bbbk H \to \Bbbk$ be a Hopf 2-cocycle such that $\sigma(x, y) = \widehat{\psi}(x, y)$ for all $x, y \in G$, as defined in* (5.2). *Denote*

$$\xi = \widehat{\psi}((u_1, 1), (1, u_2))\widehat{\psi}((1, u_2), (u_1, 1))^{-1}.$$

If $\xi = 1$ there is an isomorphism of comodule algebras

$$\mathcal{K}(W, 0, F, \psi) \simeq C(W^1, W^2, W^3, F)_\sigma.$$

If $\xi = -1$ there is an isomorphism of comodule algebras

$$\mathcal{K}(W, 0, F, \psi) \simeq \widetilde{C}(W^1, W^2, W^3, F)_\sigma.$$

Proof. One can verify that the relations that hold in $C(W^1, W^2, W^3, F)_\sigma$ are the same relations in $\mathcal{K}(W, 0, F, \psi)$. Thus there is a well-defined projection $C(W^1, W^2, W^3, F)_\sigma \twoheadrightarrow \mathcal{K}(W, 0, F, \psi)$ which is an isomorphism since both algebras have the same dimension. □

The above Proposition can be extended when the bilinear form β is not null. Let us begin by constructing a Hopf 2-cocycle in H.

Lemma 7.8. *Let $\sigma : H \otimes_\Bbbk H \to \Bbbk$ be the map defined by*

$$\sigma(x, y) = \begin{cases} \widehat{\psi}(x, y) & \text{if } x, y \in G, \\ \frac{1}{2}\widehat{\psi}(sg, th)\beta(v, w) & \text{if } x = vg,\ y = wh,\ v \in V_g,\ w \in V_t,\ g, h \in G \\ 0 & \text{otherwise.} \end{cases}$$

Then σ is a Hopf 2-cocycle. □

Let $\xi = \widehat{\psi}((u_1, 1), (1, u_2))\widehat{\psi}((1, u_2), (u_1, 1))^{-1}$ and let (W, β, F, ψ) be a compatible data.

Proposition 7.9. *If $\xi = 1$ then there is an isomorphism of comodule algebras*

$$\mathcal{K}(W, \beta, F, \psi) \simeq C(W^1, W^2, W^3, F)_\sigma.$$

If $\xi = -1$ there is an isomorphism of comodule algebras

$$\mathcal{K}(W, \beta, F, \psi) \simeq \widetilde{C}(W^1, W^2, W^3, F)_\sigma.$$

Proof. One can verify that the relations that hold in $C(W^1, W^2, W^3, F)_\sigma$ are the same relations that hold in $\mathcal{K}(W, \beta, F, \psi)$. Let us do this only for $w_2 \in W^2$, $(v, w) \in W^3$. By definition of σ we have

$$w_2 \cdot_\sigma [(v,w)] = \frac{1}{2}\beta(w_1, v)1 + \frac{1}{2}\beta(w_1, w)u + \widehat{\psi}(u_2, u_1)w_2[(v,w)],$$
$$[(v,w)] \cdot_\sigma w_2 = \frac{1}{2}\beta(v, w_1)1 + \frac{1}{2}\beta(w, w_1)u + \widehat{\psi}(u_1, u_2)[(v,w)]w_2.$$

Then

$$w_2 \cdot_\sigma [(v,w)] - [(v,w)] \cdot_\sigma w_2 = \beta(w_1, w)u.$$

Thus there is a well-defined projection $C(W^1, W^2, W^3, F)_\sigma \twoheadrightarrow \mathcal{K}(W, \beta, F, \psi)$ which is an isomorphism since both algebras have the same dimension. $\square$

Theorem 7.10. *Let $(V_1, V_2, u_1, u_2, G_1, G_2)$ be a data as in subsection 6.2 and $H = \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$. Let $\mathcal{M}$ be an indecomposable exact left* $\operatorname{Rep}(H)$*-module category. Then there is a compatible data (W, β, F, ψ) such that $\mathcal{M}$ is equivalent to the category ${}_{\mathcal{K}(W,\beta,F,\psi)}\mathcal{M}$ as* $\operatorname{Rep}(H)$*-modules.*

Proof. By Proposition 7.4 and [3, Prop. 1.20] the families ${}_{\mathcal{K}(W,\beta,F,\psi)}\mathcal{M}$ are exact indecomposable module categories over $\operatorname{Rep}(H)$.

Let $\mathcal{M}$ be an indecomposable exact $\operatorname{Rep}(H)$-module category. Then, by [3, Thm. 3.3] there exists a right H-simple left comodule algebra with trivial coinvariants (A, λ) such that $\mathcal{M} = {}_A\mathcal{M}$ as $\operatorname{Rep}(H)$-modules. Since H is coradically graded then $\operatorname{gr} A$ is a right H-simple left comodule algebra also with trivial coinvariants.

Since $H_0 = \Bbbk G$ and $A(0)$ is a left $\Bbbk G$-comodule algebra right $\Bbbk G$-simple then there exists a subgroup $F \subseteq G$ and $\psi \in Z^2(F, \Bbbk^\times)$ such that $A(0) = \Bbbk_\psi F$.

Let $\widehat{\psi} \in Z^2(G, \Bbbk^\times)$ be a 2-cocycle such that $\widehat{\psi}\,|_F = \psi$. Let $\sigma : H \otimes_\Bbbk H \to \Bbbk$ be a Hopf 2-cocycle such that $\sigma(x, y) = \widehat{\psi}(x, y)$ for all $x, y \in G$, as defined in (5.2).

By Lemma 5.5 the algebra $(\operatorname{gr} A)_{\sigma^{-1}}$ is isomorphic to a homogeneous left coideal subalgebra of $H^{[\sigma^{-1}]}$. Set $\xi = \widehat{\psi}((u_1, 1), (1, u_2))\widehat{\psi}((1, u_2), (u_1, 1))^{-1}$. Since $\xi^2 = 1$ then $\xi = \pm 1$. We shall analyze what happens in both cases.

Case $\xi = 1$. It follows from Proposition 6.2 that there is an isomorphism of Hopf algebras $H^{[\sigma^{-1}]} \simeq \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$, therefore $(\operatorname{gr} A)_{\sigma^{-1}}$ is isomorphic as a comodule algebra to a coideal subalgebra of H. Hence, from Theorem 6.6 we deduce that $(\operatorname{gr} A)_{\sigma^{-1}} = C(W^1, W^2, W^3, F)$ for some coideal subalgebra data (W^1, W^2, W^3, F). Proposition 7.7 implies that $(\operatorname{gr} A) \simeq \mathcal{K}(W, 0, F, \psi)$. Now, we have to determine all liftings of $\mathcal{K}(W, 0, F, \psi)$, that is all comodule algebras A such that $(\operatorname{gr} A) \simeq \mathcal{K}(W, 0, F, \psi)$.

For any $w_1 \in W^1$, $w_2 \in W^2$, $(v, w) \in W^3$ let be $a_{w_1}, a_{w_2}, a_{(v,w)} \in A_1$ elements such that

$$\lambda(a_{w_1}) = w_1 \otimes 1 + (u_1, 1) \otimes a_{w_1}, \quad \lambda(a_{w_2}) = w_2 \otimes 1 + (1, u_2) \otimes a_{w_2},$$

$$\lambda(a_{(v,w)}) = v \otimes 1 + wu \otimes e_u + (u_1, 1) \otimes a_{(v,w)},$$

and the class of a_w in $A(1) = A_1/A_0$ equals w. We can choose these elements so that they satisfy that

$$a_{v+w} = a_v + a_w, \quad f a_w f^{-1} = a_{f\cdot w}, \qquad \text{for all } f \in F,\ v, w \in W.$$

The proof of the existence of such elements is the same as the proof of [16, Lemma 5.5]. Then A is generated as an algebra by elements $\{a_w, f : w \in W, f \in F\}$.

For any $(i,j) \in \{(1,1),(2,2),(1,3),(3,3)\}$ take $w_i \in W^i, w_j \in W^j$. Then

$$\lambda(a_{w_i}a_{w_j} + a_{w_j}a_{w_i}) = 1 \otimes a_{w_i}a_{w_j} + a_{w_j}a_{w_i},$$

hence there exists an scalar $\beta(w_i, w_j) \in \Bbbk$ such that

$$a_{w_i}a_{w_j} + a_{w_j}a_{w_i} = \beta(w_i, w_j)\, 1.$$

If $w_1 \in W^1$, $w_2 \in W^2$ then

$$\lambda(a_{w_1}a_{w_2} - a_{w_2}a_{w_1}) = u \otimes a_{w_1}a_{w_2} - a_{w_2}a_{w_1},$$

hence there exists $\beta(w_1, w_2) \in \Bbbk$ such that

$$a_{w_1}a_{w_2} - a_{w_2}a_{w_1} = \beta(w_1, w_2)\, e_u.$$

If $u \notin F$ then $\beta(w_1, w_2) = 0$. The same is done in the case $w_2 \in W^2$, $w_3 \in W^3$. One can prove that (W, β, F, ψ) is a compatible data and there is a comodule algebra projection

$$\mathcal{K}(W, \beta, F, \psi) \twoheadrightarrow A$$

which is injective since both algebras have the same dimension.

Case $\xi = -1$**.** The proof of this case is entirely similar to the case $\xi = 1$. □

7.1. **Equivalence classes of module categories.** We shall explain when two module categories appearing in Theorem 7.10 are equivalent.

Let H be a finite-dimensional pointed Hopf algebra and $\mathcal{A}, \mathcal{A}'$ be right H-simple left A-comodule algebras with trivial coinvariants. If $g \in H$ is a group-like element we can define a new comodule algebra $\mathcal{A}^g$ on the same underlying algebra $\mathcal{A}$ with coaction given by $\lambda^g : \mathcal{A}^g \to H \otimes_{\Bbbk} \mathcal{A}^g$, $\lambda^g(a) = g a_{(-1)} g^{-1} \otimes a_{(0)}$, for all $a \in \mathcal{A}$.

Theorem 7.11. [12, Thm. 4.2] *The algebras $\mathcal{A}$, $\mathcal{A}'$ are equivariantly Morita equivalent if and only if there exists an element $g \in G(A)$ such that $\mathcal{A}' \simeq \mathcal{A}^g$ as comodule algebras.* □

Theorem 7.12. *Let $(V_1, V_2, u_1, u_2, G_1, G_2)$ be a data as in subsection 6.2 and set $H = \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$. Let (W, β, F, ψ), (U, β', F', ψ') be two compatible data. The module categories ${}_{\mathcal{K}(W,\beta,F,\psi)}\mathcal{M}$, ${}_{\mathcal{K}(U,\beta',F',\psi')}\mathcal{M}$ are equivalent if and only if there exists $g \in G$ such that*

$$W^1 = g \cdot U^1, \quad W^2 = g \cdot U^2, \quad W^3 = g \cdot U^3, \quad \beta' = g \cdot \beta, \quad F' = gFg^{-1}, \quad \psi' = \psi^g.$$

Here $g \cdot \beta(v, w) = \beta(g^{-1} \cdot v, g^{-1} \cdot w)$ for all $v, w \in U$.

Proof. Let us prove that if $\mathcal{K}(W,\beta,F,\psi)$ and $\mathcal{K}(W',\beta',F',\psi')$ are isomorphic as H-comodule algebras then $W=W'$, $\beta=\beta'$, $F=F'$ and $\psi=\psi'$. Let $\vartheta:\mathcal{K}(W,\beta,F,\psi)\to\mathcal{K}(W',\beta',F',\psi')$ be an isomorphism of H-comodule algebras, then for any $f\in F$ we have that

$$f\otimes\vartheta(e_f)=\lambda(\vartheta(e_f)).$$

This implies that $\vartheta(e_f)\in\mathcal{K}(W',\beta',F',\psi')_0=\Bbbk F'$ and has no other possibility than being equal to e_f. Hence $F\subseteq F'$. The other inclusion can be proven using the inverse of ϑ. Since ϑ is an algebra morphism we deduce that $\psi=\psi'$.

It is not difficult to prove that for any $i=1,2,3$ we have that $\vartheta(W^i)\subseteq U^i$. Since ϑ is an isomorphism then $W^i=U^i$ for any $i=1,2,3$. Since ϑ is an algebra morphism the bilinear forms β,β' must be equal.

For any $g\in G$ there is an isomorphism of comodule algebras

$$\mathcal{K}(W,\beta,F,\psi)^g\simeq\mathcal{K}(g\cdot W,g\cdot\beta,gFg^{-1},\psi^g).$$

Indeed, the algebra map $\theta:\mathcal{K}(W,\beta,F,\psi)^g\to\mathcal{K}(g\cdot W,g\cdot\beta,gFg^{-1},\psi^g)$ determined by

$$\theta(w)=g\cdot w,\quad \theta(e_f)=e_{gfg^{-1}},$$

for all $w\in W$, $f\in F$, is an isomorphism of comodule algebras. The proof of the Theorem follows now from Theorem 7.11. □

8. The Brauer-Picard group of supergroup algebras

The *Brauer-Picard groupoid* [9] $\underline{\underline{\mathrm{BrPic}}}$ is the 3-groupoid whose objects are finite tensor categories, 1-morphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$ are invertible exact $(\mathcal{C}_1,\mathcal{C}_2)$-bimodule categories, 2-morphisms are equivalences of such bimodule categories, and 3-morphisms are isomorphisms of such equivalences. Forgetting the 3-morphisms and the 2-morphisms and identifying 1-morphisms one obtains the groupoid BrPic. For a fixed tensor category $\mathcal{C}$, the group $\mathrm{BrPic}(\mathcal{C})$ consists of equivalence classes of invertible exact $\mathcal{C}$-bimodule categories and it is called the *Brauer-Picard group of $\mathcal{C}$*.

In this section G will denote a finite Abelian group, V is a finite-dimensional G-module, and $u\in G$ is an element of order 2 such that it acts on V as -1. Also $H=\mathcal{A}(V,u,G)$.

8.1. The Brauer-Picard group of group algebras.

Let us recall the results obtained in [9] on the computation of the Brauer-Picard group of the category of representations of a finite Abelian group.

Definition 8.1. Let G be a finite Abelian group. The group $O(G\oplus\widehat{G})$ consists of group isomorphisms $\alpha:G\oplus\widehat{G}\to G\oplus\widehat{G}$ such that $\langle\alpha_2(g,\chi),\alpha_1(g,\chi)\rangle=\langle\chi,g\rangle$ for all $g\in G,\chi\in\widehat{G}$. Here $\alpha(g,\chi)=(\alpha_1(g,\chi),\alpha_2(g,\chi))$.

Theorem 8.2. [9, Corollary 1.2] *There is an isomorphism of groups*

$$\mathrm{BrPic}(\mathrm{Rep}(G))\simeq O(G\oplus\widehat{G}).$$ □

Let us explain how to obtain invertible bimodule categories from elements in $O(G \oplus \widehat{G})$. Let $\alpha \in O(G \oplus \widehat{G})$ and define $U_\alpha \subseteq G \times G$ the subgroup

$$U_\alpha = \{(\alpha_1(g,\chi), g) : g \in G, \chi \in \widehat{G}\},$$

and the 2-cocycle $\psi_\alpha : U_\alpha \times U_\alpha \to \Bbbk^\times$ defined by

$$\psi_\alpha((\alpha_1(g,\chi), g), (\alpha_1(h,\xi), h)) = \langle \alpha_2(g,\chi)^{-1}, \alpha_1(h,\xi)\rangle \langle \chi, h\rangle.$$

It was proved in [9] that the bimodule categories ${}_{\Bbbk_{\psi_\alpha} U_\alpha}\mathcal{M}$ are invertible and any invertible bimodule category is equivalent to one of this form. Note that $U_{\mathrm{id}} = \mathrm{diag}(G), \psi_{\mathrm{id}} = 1$.

Example 8.3. If $G = \mathbb{Z}_p$ for some prime $p \in \mathbb{N}$ then $O(G \oplus \widehat{G})$ is isomorphic to the dihedral group $\mathbb{D}_{2(p-1)}$. In particular if $p = 2$ then $O(\mathbb{Z}_2 \oplus \widehat{\mathbb{Z}_2}) \simeq \mathbb{Z}_2$. The only non-trivial element in $O(\mathbb{Z}_2 \oplus \widehat{\mathbb{Z}_2})$ is $\gamma : \mathbb{Z}_2 \oplus \widehat{\mathbb{Z}_2} \to \mathbb{Z}_2 \oplus \widehat{\mathbb{Z}_2}$ given by

$$\gamma(u^i, \chi^j) = (u^j, \chi^i), \tag{8.1}$$

for $i, j = 0, 1$. Here u is the generator of $\mathbb{Z}_2$ and χ is the generator of $\widehat{\mathbb{Z}_2}$.

8.2. **Families of invertible bimodule categories.** In this section we present families of invertible $\mathrm{Rep}(H)$-bimodule categories.

Definition 8.4. We shall denote by $\mathcal{R}(V, u, G)$ the set of collections (W, β, α), where

(i) $W \subseteq V \oplus V$ is a subspace such that $W \cap V \oplus 0 = 0 = W \cap 0 \oplus V$,
(ii) $\alpha \in O(G \oplus \widehat{G})$ is an isomorphism such that $(u, u) \in U_\alpha$,
(iii) W is invariant under the action of U_α,
(iv) $\beta : W \times W \to \Bbbk$ is a symmetric bilinear form invariant under the action of U_α.

If (W, β, α), $(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha})$ are elements in $\mathcal{R}(V, u, G)$ we define

$$(W, \beta, \alpha) \bullet (\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha}) = (W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha\widetilde{\alpha}), \tag{8.2}$$

where $W \bullet \widetilde{W}$ is the subspace of $V \oplus V$ consisting of elements (v_1, w_1) such that there exists a (necessarily unique) $v_2 \in V$ such that $(v_1, v_2) \in W$, $(v_2, w_1) \in \widetilde{W}$. The bilinear form $\beta \bullet \widetilde{\beta}$ is defined by

$$\beta \bullet \widetilde{\beta}((v_1, w_1), (v_1', w_1')) = \beta((v_1, v_2), (v_1', v_2')) + \widetilde{\beta}((v_2, w_1), (v_2', w_1')),$$

where $v_2, v_2' \in V_2$ are the unique elements such that $(v_1, v_2), (v_1', v_2') \in W$ and (v_2, w_1), $(v_2', w_1') \in \widetilde{W}$. The action of $U_{\alpha\widetilde{\alpha}}$ on $W \bullet \widetilde{W}$ is given as follows. If $g \in G, \chi \in \widehat{G}$, $(v, w) \in W \bullet \widetilde{W}$ then

$$(\alpha_1(\widetilde{\alpha}(g,\chi)), g) \cdot (v, w) = (\alpha_1(\widetilde{\alpha}(g,\chi)) \cdot v, g \cdot w).$$

Lemma 8.5. *If* (W, β, α), $(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha})$ *then* $(W, \beta, \alpha) \bullet (\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha}) \in \mathcal{R}(V, u, G)$.

Proof. We will only prove that the bilinear form $\beta \bullet \widetilde{\beta}$ is invariant under the action of $U_{\alpha\widetilde{\alpha}}$. The other properties are straightforward. Let $(v_1, w_1), (v_1', w_1') \in W \bullet \widetilde{W}$ and $(f, g) \in U_{\alpha\widetilde{\alpha}}$, then $\beta \bullet \widetilde{\beta}((f, g) \cdot (v_1, w_1), (f, g) \cdot (v_1', w_1'))$ is equal to

$$\begin{aligned}
&= \beta \bullet \widetilde{\beta}((f \cdot v_1, g \cdot w_1), (f \cdot v_1', g \cdot w_1')) \\
&= \beta((f \cdot v_1, x \cdot v_2), (f \cdot v_1', x \cdot v_2')) + \widetilde{\beta}((x \cdot v_2, g \cdot w_1), (x \cdot v_2', g \cdot w_1')) \\
&= \beta((v_1, v_2), (v_1', v_2')) + \widetilde{\beta}((v_2, w_1), (v_2', w_1')) \\
&= \beta \bullet \widetilde{\beta}((v_1, w_1), (v_1', w_1')).
\end{aligned}$$

In the above equalities the element $x \in G$ is the unique such that $(f, x) \in U_\alpha$ and $(x, g) \in U_{\widetilde{\alpha}}$, and $v_2, v_2' \in V_2$ are the unique elements such that $(v_1, v_2), (v_1', v_2') \in W$ and (v_2, w_1), $(v_2', w_1') \in \widetilde{W}$. □

Definition 8.6. We say that $(W, \beta, \alpha) \sim (\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha})$ if there exists an element $g \in G \times G$ such that

$$\widetilde{W} = g \cdot W, \quad \widetilde{\beta} = g \cdot \beta, \alpha = \widetilde{\alpha}.$$

If $(W, \beta, U_\alpha, \psi_\alpha)$ is a compatible family for some $\alpha \in O(G \oplus \widehat{G})$ we shall denote

$$\mathcal{K}(W, \beta, \alpha) = \mathcal{K}(W, \beta, U_\alpha, \psi_\alpha), \quad \mathcal{L}(W, \beta, \alpha) = \mathcal{L}(W, \beta, U_\alpha, \psi_\alpha).$$

Theorem 8.7. *Let $(W, \beta, \alpha) \in \mathcal{R}(V, u, G)$ such that there exists $(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha}) \in \mathcal{R}(V, u, G)$ such that*

$$(W, \beta, \alpha) \bullet (\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha}) \sim (\mathrm{diag}(V), 0, \mathrm{id}), \tag{8.3}$$

$$(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha}) \bullet (W, \beta, \alpha) \sim (\mathrm{diag}(V), 0, \mathrm{id}). \tag{8.4}$$

Then the $\mathrm{Rep}(H)$*-bimodule category* ${}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M}$ *is invertible.*

Proof. The proof is a (more complicated) version of the proof of the fundamental theorem for Hopf modules [18]. If $L = \mathcal{L}(W, \beta, \alpha), K = \mathcal{L}(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha})$, we shall prove that the categories $\mathcal{M}(H, H, K, \overline{L})$, ${}_{\mathcal{L}(\mathrm{diag}(V),0,\mathrm{id})}\mathcal{M}$ are equivalent as bimodule categories.

Let us fix some notation. If $(v_1, w_1) \in W \bullet \widetilde{W}$ then there exists a unique $v_2 \in V_2$ such that $(v_1, v_2) \in W$, $(v_2, w_1) \in \widetilde{W}$. We shall denote

$$\iota_1(v_1, w_1) = (v_1, v_2), \quad \iota_2(v_1, w_1) = (v_2, w_1).$$

Analogously if $(v_1, w_1) \in \widetilde{W} \bullet W$ there exists a unique $v_2 \in V_1$ such that $(v_1, v_2) \in \widetilde{W}$ and $(v_2, w_1) \in W$. We shall denote

$$\widetilde{\iota}_1(v_1, w_1) = (v_1, v_2), \quad \widetilde{\iota}_2(v_1, w_1) = (v_2, w_1).$$

From (8.3), (8.4) it follows that there are elements $(a, b), (g, h) \in G \times G$ such that

(i) $W \bullet \widetilde{W} = (g, h) \cdot \mathrm{diag}(V)$,
(ii) $\widetilde{W} \bullet W = (a, b) \cdot \mathrm{diag}(V)$.

Denote $S = \mathcal{L}(\mathrm{diag}(V), 0, \mathrm{id})$. Let

$$\phi : S \to L \otimes_{\Bbbk} K, \quad \overline{\phi} : S \to K \otimes_{\Bbbk} L,$$

be the algebra morphisms determined as follows. If $w \in V$, $f \in G$ then

$$\begin{aligned}\phi(w, w) &= \iota_1(g \cdot w, h \cdot w) \otimes 1 + e_{(u,u)} \otimes \iota_2(g \cdot w, h \cdot w),\\ \phi(e_{(f,f)}) &= e_{(f,\widetilde{\alpha}(f,1))} \otimes e_{(\widetilde{\alpha}(f,1),f)}.\end{aligned}$$

If $v \in V$, $x \in G$ then

$$\begin{aligned}\overline{\phi}(v, v) &= \widetilde{\iota}_1(a \cdot v, b \cdot v) \otimes 1 + e_{(u,u)} \otimes \widetilde{\iota}_2(a \cdot v, b \cdot v),\\ \overline{\phi}(e_{(x,x)}) &= e_{(x,\alpha(x,1))} \otimes e_{(\alpha(x,1),x)}.\end{aligned}$$

Claim 8.1. *The maps $\phi, \overline{\phi}$ are well-defined.*

Proof of Claim. One should prove that for all $v, w \in V$, $f, g \in G \times G$

$$\phi(w, w)\phi(v, v) + \phi(v, v)\phi(w, w) = 0, \tag{8.5}$$

$$\phi(e_f)\phi(e_g) = \phi(e_{fg}), \tag{8.6}$$

$$\phi(e_f)\phi(w, w) = \phi(f \cdot (w, w))\phi(e_f). \tag{8.7}$$

The verification of these equalities is straightforward. The same equations hold for $\overline{\phi}$. ☐

Let us recall the isomorphism of $\mathcal{A}(V, V, u, u, G, G)$-comodule algebras $\sigma : \mathrm{diag}(H) \to S$ presented in the proof of Lemma 7.5. We shall use the notation

$$\phi(s) = \phi^1(s) \otimes \phi^2(s), \quad \overline{\phi}(t) = \overline{\phi}^1(t) \otimes \overline{\phi}^2(t),$$

omitting the summation symbol, for all $s, t \in S$.

Claim 8.2. *If $s, t \in S$ then*

$$\pi_2(\overline{\phi}^1(t)_{-1})\mathcal{S}^{-1}(\pi_2(\overline{\phi}^2(t)_{-1})) \otimes \overline{\phi}^1(t)_0 \otimes \overline{\phi}^2(t)_0 = 1 \otimes \overline{\phi}(t) \tag{8.8}$$

$$\pi_2(\phi^2(s)_{-1})\mathcal{S}^{-1}(\pi_2(\phi^1(s)_{-1})) \otimes \phi^1(s)_0 \otimes \phi^2(s)_0 = 1 \otimes \phi(s). \tag{8.9}$$

The proof follows by verifying that both equalities hold for the generators of the algebra S and using that both maps $\phi, \overline{\phi}$ are algebra morphisms.

If $M \in \mathcal{M}(H, H, K, \overline{L})$ define $\pi_M : M \to M^{\mathrm{co}\, H}$ the map

$$\pi_M(m) = \overline{\phi}^1(\sigma(\mathcal{S}(m_{(-1)}))) \cdot m_{(0)} \cdot \overline{\phi}^2(\sigma(\mathcal{S}(m_{(-1)}))).$$

It follows from (8.8) that the image of π_M is indeed inside $M^{\mathrm{co}\, H}$. The space $M^{\mathrm{co}\, H}$ has a left S-action given by

$$s \cdot m = \phi^2(s) \cdot m \cdot \phi^1(s),$$

for all $s \in S_1$, $m \in M^{\mathrm{co}\, H}$. It follows from (8.9) that this action is well-defined, that is, if $s \in S$, $m \in M^{\mathrm{co}\, H}$ then $s \cdot m \in M^{\mathrm{co}\, H}$.

Let $\mathcal{G} : \mathcal{M}(H, H, K, \overline{L}) \to {}_S\mathcal{M}$ and $\mathcal{F} : {}_S\mathcal{M} \to \mathcal{M}(H, H, K, \overline{L})$ be the functors defined as follows. If $M \in \mathcal{M}(H, H, K, \overline{L})$, $N \in {}_S\mathcal{M}$ then

$$\mathcal{F}(N) = (L \otimes_{\Bbbk} K) \otimes_S N, \qquad \mathcal{G}(M) = M^{\mathrm{co}\, H}.$$

The structure of right S-module on $L\otimes_{\Bbbk}K$ is given via ϕ. Both functors are bimodule functors, see [17, Prop. 3.7]. These functors are in fact the same (up to some minor modifications) functors described in Section 4.1. For any $M\in\mathcal{M}(H,H,K,\overline{L})$ define

$$\alpha_M : M\to (L\otimes_{\Bbbk}K)\otimes_S M^{\mathrm{co}\,H},\quad \beta_M : (L\otimes_{\Bbbk}K)\otimes_S M^{\mathrm{co}\,H}\to M,$$
$$\alpha_M(m)=\overline{\phi}(\sigma(m_{(-1)}))\otimes\pi_M(m_{(0)}),\quad \beta_M(l\otimes k\otimes m)=k\cdot m\cdot l,$$

for all $m\in M^{\mathrm{co}\,H}$, $l\in L$, $k\in K$.

Claim 8.3. *The maps α_M, β_M are inverse of each other.*

Proof of claim. Let $m\in M$ then $\beta_M\circ\alpha_M(m)$ is equal to

$$\begin{aligned}
&=\overline{\phi}^1(\sigma(m_{(-1)}))\cdot\pi_M(m_{(0)})\cdot\overline{\phi}^2(\sigma(m_{(-1)}))\\
&=\overline{\phi}^1(\sigma(m_{(-1)})\sigma(\mathcal{S}(m_{(0)(-1)})))\cdot m_{(0)(0)}\cdot\overline{\phi}^2(\sigma(m_{(-1)})\sigma(\mathcal{S}(m_{(0)(-1)})))\\
&=\epsilon(m_{(-1)})m_{(0)}=m.
\end{aligned}$$

Let $m\in M^{\mathrm{co}\,H}$, $l\in L$, $k\in K$. Then $\alpha_M\circ\beta_M(l\otimes k\otimes m)$ is equal to

$$\begin{aligned}
&=\alpha_M(k\cdot m\cdot l)\\
&=\overline{\phi}\sigma\pi_2(k_{(-1)}\mathcal{S}^{-1}(l_{(-1)}))\otimes\pi_M(k_{(0)}\cdot m\cdot l_{(0)})\\
&=\overline{\phi}\sigma\pi_2(k_{(-1)}\mathcal{S}^{-1}(l_{(-1)}))\\
&\quad\otimes\overline{\phi}^1\sigma\pi_2(k_{(0)(-1)}\mathcal{S}^{-1}(l_{(0)(-1)}))k_{(0)(0)}\cdot m\cdot l_{(0)(0)}\overline{\phi}^2\sigma\pi_2(k_{(0)(-1)}\mathcal{S}^{-1}(l_{(0)(-1)})).
\end{aligned}$$

Now, to prove that $\alpha_M\circ\beta_M(l\otimes k\otimes m)=(l\otimes k\otimes m)$ it is enough to prove that

$$\overline{\phi}\sigma\pi_2(k_{(-1)})\otimes\overline{\phi}^1\sigma\pi_2(k_{(0)(-1)})k_{(0)(0)}\otimes\overline{\phi}^2\sigma\pi_2(k_{(0)(-1)})=k\otimes 1\otimes 1\otimes 1, \tag{8.10}$$

$$\overline{\phi}\sigma\pi_2(\mathcal{S}^{-1}(l_{(-1)}))\otimes\overline{\phi}^1\sigma\pi_2(l_{(0)(-1)})\otimes l_{(0)(0)}\overline{\phi}^2\sigma\pi_2(\mathcal{S}^{-1}(l_{(0)(-1)}))=1\otimes l\otimes 1\otimes 1. \tag{8.11}$$

Since $\overline{\phi}\sigma\pi_2$ is an algebra map, equations (8.10) and (8.11) can be verified on the generators of the algebras L and K. □

In conclusion we have that $\mathcal{F}\mathcal{G}=\mathrm{Id}$. Let us prove that $\mathcal{G}\mathcal{F}=\mathrm{Id}$. For any $N\in{}_S\mathcal{M}$ we have an inclusion

$$N\hookrightarrow\mathcal{G}(\mathcal{F}(N)),\quad n\mapsto 1\otimes 1\otimes n,$$

for all $n\in N$. Let $\Psi:{}_S\mathcal{M}\to vect_{\Bbbk}$ be the functor defined by $\Psi(N)=\mathcal{G}(\mathcal{F}(N))/N$ for all $N\in{}_S\mathcal{M}$. The functor Ψ is a $\mathrm{Rep}(H)$-module functor. Indeed, define $c_{X,N}:\Psi(X\otimes_{\Bbbk}N)\to X\otimes_{\Bbbk}\Psi(N)$ by

$$c_{X,N}(l\otimes k\otimes x\otimes n)=\pi_1(l_{(-1)})\cdot x\otimes l_{(0)}\otimes k\otimes n,$$

for all $X\in\mathrm{Rep}(H)$, $N\in{}_1\mathcal{M}$, $l\in L$, $k\in K$, $x\in X$, $n\in N$. It follows straightforwardly that (Ψ,c) is a module functor, thus it is exact. The full subcategory $\mathcal{N}$ of objects such that $\Psi(N)=0$ is a submodule category of ${}_S\mathcal{M}$. Since $\Psi(S)=0$ and the category ${}_S\mathcal{M}$ is indecomposable, then $\mathcal{N}={}_S\mathcal{M}$, which implies that $\mathcal{G}\mathcal{F}=\mathrm{Id}$. □

Theorem 8.8. *If $\alpha, \widetilde{\alpha}$ are elements in $O(G \oplus \widehat{G})$ and $K = \mathcal{L}(\mathrm{diag}(V), 0, \alpha)$, $L = \mathcal{L}(\mathrm{diag}(V), 0, \widetilde{\alpha})$, then there is an equivalence of bimodule categories*

$$\mathcal{M}(H, H, K, \overline{L}) \simeq {}_{\mathcal{L}(\mathrm{diag}(V),0,\alpha\widetilde{\alpha})}\mathcal{M}.$$

Proof. The proof of Theorem 8.7 applies *mutatis mutandis* to this case. ☐

8.3. **The Brauer-Picard group of supergroup algebras.** The comodule algebras $\mathcal{L}(W, \beta, \alpha)$ will be the relevant ones when computing the Brauer-Picard group for the representations categories of supergroup algebras. In Theorem 8.7 we have seen that the bimodule categories ${}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M}$ are invertible if the compatible data (W, β, α) is invertible in some sense. We must prove now that these categories are the only invertible bimodule categories and we have to describe the tensor product between them. In view of Theorem 4.2 we need first to investigate the cotensor product of two such comodule algebras.

Let (W, β, α), $(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha})$ be compatible data with (V, u, G). We shall further assume that W and $\widetilde{W}$ have decompositions

$$W = 0 \oplus 0 \oplus W^3, \quad \widetilde{W} = 0 \oplus 0 \oplus \widetilde{W}^3.$$

Let $L = \mathcal{L}(W, \beta, \alpha)$, $K = \mathcal{L}(\widetilde{W}, \widetilde{\beta}, \widetilde{\alpha})$. If $(v_1, w_1) \in W \bullet \widetilde{W}$ then there exists a unique $v_2 \in V_2$ such that $(v_1, v_2) \in W$, $(v_2, w_1) \in \widetilde{W}$. We shall denote

$$\iota_1(v_1, w_1) = (v_1, v_2), \quad \iota_2(v_1, w_1) = (v_2, w_1).$$

Analogously if $(v_1, w_1) \in \widetilde{W} \bullet W$ there exists a unique $v_2 \in V$ such that $(v_1, v_2) \in \widetilde{W}$ and $(v_2, w_1) \in W$. We shall denote

$$\widetilde{\iota}_1(v_1, w_1) = (v_1, v_2), \quad \widetilde{\iota}_2(v_1, w_1) = (v_2, w_1).$$

Let $p_1, p_2 : V \oplus V \to V$ the canonical projections, so $p_1(v, w) = v$ and $p_2(v, w) = w$ for all $(v, w) \in V \oplus V$. Abusing the notation we shall also denote by $p_1, p_2 : G \times G \to G$ the canonical projections.

Lemma 8.9. *Let $\{(w_i^1, w_i^2)\}_{i=1}^t$ be a basis of $W \bullet \widetilde{W} \subseteq V \oplus V$. There exists a basis $\{v_i\}_{i=1}^n$ of W and a basis $\{w_i\}_{i=1}^m$ of $\widetilde{W}$ such that $t \leq n$, $t \leq m$ and $p_1(v_i) = w_i^1$, $p_2(w_i) = w_i^2$ for any $i = 1, \ldots, t$.*

Proof. For any $i = 1, \ldots, t$ there exists $t_i \in V$ such that $(w_i^1, t_i) \in W$, $(t_i, w_i^2) \in \widetilde{W}$. The sets $\{(w_i^1, t_i)\}_{i=1}^t$, $\{(t_i, w_i^2)\}_{i=1}^t$ are linearly independent, thus we can extend both set to a basis in their corresponding spaces. ☐

The right H-comodule structure described in (4.2) will be denoted by $\lambda_r : L \to L \otimes_{\Bbbk} H$ and if $x \in W$ then

$$\lambda_r(x) = e_{(u,u)} \otimes p_2(x) + x \otimes 1. \tag{8.12}$$

The left H-comodule structure described in (4.3) will be denoted by $\lambda_l : K \to H \otimes_{\Bbbk} K$ and if $y \in \widetilde{W}$ then

$$\lambda_l(y) = p_2(y) \otimes 1 + u \otimes y. \tag{8.13}$$

The proof of the next Lemma can be done using an inductive argument.

Lemma 8.10. *If $x = x_1 \dots x_n e_f \in L$ and $y = y_1 \dots y_m e_h \in K$ are elements such that $x_1, \dots, x_n \in W$, $y_1, \dots, y_m \in \widetilde{W}$ $f \in U_\alpha$, $h \in U_{\widetilde{\alpha}}$ then*

$$\lambda_r(x) = \sum_{\substack{\epsilon_i, \delta_i \in \{0,1\} \\ \epsilon_i + \delta_i = 1}} \alpha^\epsilon_\delta \; x_1^{\epsilon_1} \dots x_n^{\epsilon_n} e_u^{\delta_1 + \dots + \delta_n} e_f \otimes p_2(x_1)^{\delta_1} \dots p_2(x_n)^{\delta_n} p_2(f), \tag{8.14}$$

$$\lambda_l(y) = \sum_{\substack{\epsilon_i, \delta_i \in \{0,1\} \\ \epsilon_i + \delta_i = 1}} \zeta^\epsilon_\delta \; p_1(y_1)^{\epsilon_1} \dots p_m(y_m)^{\epsilon_m} u^{\epsilon_1 + \dots + \epsilon_m} p_2(h) \otimes y_1^{\delta_1} \dots y_m^{\delta_m} e_h, \tag{8.15}$$

where all coefficients $\alpha^\epsilon_\delta, \zeta^\epsilon_\delta \in \Bbbk$ are not null. □

Proposition 8.11. *Assume that $\widetilde{\alpha} = \mathrm{id}$. Then there is an isomorphism of left $H \otimes_\Bbbk H^{\mathrm{cop}}$-comodule algebras*

$$\mathcal{L}(W, \beta, \alpha) \square_H \mathcal{L}(\widetilde{W}, \widetilde{\beta}, \mathrm{id}) \simeq \mathcal{L}(W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha). \tag{8.16}$$

Proof. Note that $\mathcal{L}(\widetilde{W}, \widetilde{\beta}, \mathrm{id})_0 = \mathrm{diag}(G)$. Let $\{x_1, \dots, x_n\}$ be a basis of W and $\{y_1, \dots, y_m\}$ be a basis of $\widetilde{W}$ such that they are extensions of a basis of $W \bullet \widetilde{W}$ in the sense of Lemma 8.9. Without loss of generality we can assume that $m \leq n$.

For any $0 \leq s \leq n, 0 \leq t \leq m$ define $L(s)$ the subspace of L generated by elements of the form

$$x_1^{\epsilon_1} \dots x_n^{\epsilon_n} e_f, \quad \text{where } \epsilon_i = 0, 1, \;\; \epsilon_1 + \dots + \epsilon_n = s, \;\; f \in U_\alpha.$$

Analogously, define $K(t)$ the subspace of K gernerated by elements of the form

$$y_1^{\delta_1} \dots y_m^{\delta_m} e_f, \quad \text{where } \delta_j = 0, 1, \;\; \delta_1 + \dots + \delta_m = t, \;\; f \in \mathrm{diag}(G).$$

Then $L = \oplus_{s=0}^n L(s)$ and $K = \oplus_{t=0}^m K(t)$.

For any $i = 1, \dots, m$ set $w_i = (p_1(x_i), p_2(y_i))$, $\pi : K \to \Bbbk 1$ the canonical projection, and define $S \subseteq F$ the subset of elements (f_1, f_2) such that there exists $(f_2, g) \in \widetilde{F}$. We shall denote by $p : \Bbbk F \to \Bbbk F \bullet \widetilde{F}$ the linear map determined by

$$p(e_{(f_1, f_2)}) = \begin{cases} 0 & \text{if } (f_1, f_2) \notin S \\ e_{(f_1, g)} & \text{if } (f_2, g) \in \mathrm{diag}(G). \end{cases}$$

By the assumptions on F and $\widetilde{F}$ the map p is well-defined. Define the map $\theta : L \square_H K \to \mathcal{L}(W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha)$ as follows. If $d \in \mathbb{N}$ and $z \in L \square_H K$ is an element of the form

$$z = \sum_{\substack{a_1 + \dots + a_m + b_1 + \dots + b_n = d \\ f \in U_\alpha,\, h \in \mathrm{diag}(G)}} \beta^{a,f}_{b,h} \; x_1^{a_1} \dots x_n^{a_n} e_f \otimes y_1^{b_1} \dots y_m^{b_m} e_h, \tag{8.17}$$

then

$$\theta(z) = \sum_{\substack{a_1 + \dots + a_m = d \\ f \in U_\alpha}} \beta^{a,f}_{0,1} \; w_1^{a_1} \dots w_m^{a_m} e_{(p_1(f), p_2(h))}.$$

From now on we shall write $a + b = d$ when we mean that $a_1 + \dots + a_m + b_1 + \dots + b_n = d$.

Claim 8.4. *The map θ is a well-defined injective linear map. In particular we have that* $\dim(L\square_H K) \leq \dim(\mathcal{L}(W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha))$.

Proof of claim. We must prove that θ is well-defined in $L\square_H K$ and that it is injective. Let us prove the first. Let be $0 \leq d \leq n+m$ and $z \in L\square_H K \cap \oplus_{s=0}^{n+m} L(s)\otimes_{\Bbbk} K(d-s)$ a non-zero element, then there are scalars $\beta_{b,h}^{a,f} \in \Bbbk$ such that

$$z = \sum_{\substack{a+b=d \\ f\in U_\alpha, h\in \mathrm{diag}(G)}} \beta_{b,h}^{a,f}\ x_1^{a_1}\dots x_n^{a_n} e_f \otimes y_1^{b_1}\dots y_m^{b_m} e_h. \tag{8.18}$$

Using (8.14) and (8.15) one gets that

$$\sum_{\substack{a+b=d \\ \epsilon_i+\delta_i=a_i \\ f\in U_\alpha, h\in \mathrm{diag}(G)}} \beta_{b,h}^{a,f} \alpha_\delta^\epsilon\ x_1^{\epsilon_1}\dots x_n^{\epsilon_n} e_u^\epsilon e_f \otimes p_2(x_1)^{\delta_1}\dots p_2(x_n)^{\delta_n} p_2(f) \otimes y_1^{b_1}\dots y_m^{b_m} e_h \tag{8.19}$$

equals

$$\sum_{\substack{a+b=d \\ \epsilon_i+\delta_i=b_i \\ f\in U_\alpha, h\in \mathrm{diag}(G)}} \beta_{b,h}^{a,f} \zeta_\delta^\epsilon\ x_1^{a_1}\dots x_n^{a_n} e_f \otimes p_1(y_1)^{\epsilon_1}\dots p_m(y_m)^{\epsilon_m} u_2^\epsilon p_2(h) \otimes y_1^{\delta_1}\dots y_m^{\delta_m} e_h. \tag{8.20}$$

Since $z \neq 0$ there exists some $\beta_{b,h}^{a,f} \neq 0$. Define

$$I(z) = \{1 \leq i \leq n : \text{there exists } \beta_{b',h}^{a',f} \neq 0 \text{ and } a'_i = 1\}.$$

Let us assume that $1 \in I(z)$, thus there exists some $\beta_{b',h}^{a',f} \neq 0$ where $a'_1 = 1$. The next argument does not depend on this choice but it simplifies the notation.

Comparing elements (8.19) and (8.20) we conclude, perhaps after reordering the elements of the basis $\{y_1, \dots, y_m\}$, that

$$\sum_{\substack{a+b=d \\ f\in U_\alpha, h\in \mathrm{diag}(G)}} \beta_{b,h}^{a,f} \alpha_{(1,0,\dots,0)}^{(0,a_2,\dots,a_n)}\ x_2^{a_2}\dots x_n^{a_n} e_u^a e_f \otimes p_2(x_1) p_2(f) \otimes y_1^{b_1}\dots y_m^{b_m} e_h \tag{8.21}$$

must be equal to

$$\sum_{\substack{a+b=d \\ f\in U_\alpha, h\in \mathrm{diag}(G)}} \beta_{b,h}^{a,f} \zeta_{(1,0,\dots,0)}^{(0,a_2,\dots,a_n)}\ x_1^{a_1}\dots x_n^{a_n} e_f \otimes p_1(y_1) u_2 p_2(h) \otimes y_2^{b_2}\dots y_m^{b_m} e_h. \tag{8.22}$$

Since $\beta_{b',h}^{a',f} \neq 0$ then $p_1(y_1) = p_2(x_1)$ and $p_2(f) = up_2(h) = p_2((u,u)h)$.

Let $\lambda_1 : L\otimes_{\Bbbk} K \to H\otimes_{\Bbbk} H^{\mathrm{cop}} \otimes_{\Bbbk} L\otimes_{\Bbbk} K$ be the coaction given in (4.4), that is if $l\otimes k \in L\otimes_{\Bbbk} K$ then

$$\lambda(l\otimes k) = \pi_1(l_{(-1)}) \otimes \pi_1(k_{(-1)}) \otimes l_{(0)} \otimes k_{(0)}.$$

With this coaction $L\otimes_{\Bbbk} K$ is a comodule algebra and has $L\square_H K$ is a subcomodule algebra. Taking $H' = H\otimes_{\Bbbk} H^{\mathrm{cop}}$, $A = L\otimes_{\Bbbk} K$ and $B = L\square_H K$ we are under the hypothesis of Lemma 5.1. This implies that any element $z \in L\square_H K$ can be written as

$$z = \sum_{d=0}^{n+m} z_d,$$

where $z_d \in L\square_H K \cap \oplus_{s=0}^{n+m} L(s)\otimes_{\Bbbk} K(d-s)$. Let us prove now that θ is injective. Assume that z is an element as in (8.17) such that $\theta(z) = 0$ and $z \neq 0$. Hence any $\beta_{0,1}^{a,f} = 0$ for any a such that $a_1 + \cdots + a_m = d$. Since $z \neq 0$ there exists at least one coefficient $\beta_{b',h}^{a',f} \neq 0$. Let us compare the coefficient of the term

$$x_1^{a'_1} \dots x_n^{a'_n} e_f \otimes p_1(y_1)^{b'_1} \dots p_m(y_m)^{b'_m} u^{b'_1 + \dots b'_m} \otimes 1 \tag{8.23}$$

in equations (8.19), (8.20). The coefficient of the term (8.23) in the summand (8.19) is $\beta_{0,1}^{\widetilde{a},f} \alpha_{b'}^{a'}$, for some $\widetilde{a} = (\widetilde{a}_1, \dots, \widetilde{a}_n)$ such that $\widetilde{a}_1 + \cdots + \widetilde{a}_n = d$, and in the summand (8.20) is $\beta_{b',1}^{a',f} \zeta_0^{b'}$. Thus $\beta_{b',1}^{a',f} \zeta_0^{b'} = \beta_{0,1}^{\widetilde{a},f} \alpha_{b'}^{a'} = 0$, whence $\beta_{b',h}^{a',f} = 0$, which is a contradiction, thus θ is injective. This finishes the proof of the claim. □

Define $\phi : \mathcal{L}(W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha) \to L\square_H K$ the algebra map determined as follows. If $w \in W \bullet \widetilde{W}$ then

$$\phi(w) = \iota_1(w) \otimes 1 + e_u \otimes \iota_2(w),$$

and if $(f, g) \in U_\alpha$ then

$$\phi(e_{(f,g)}) = e_{(f,g)} \otimes e_{(g,g)}.$$

The map ϕ extends to a comodule algebra morphism and the image is contained in $L\square_H K$. To prove that ϕ is well-defined one should verify that

$$\phi(w)\phi(v) + \phi(v)\phi(w) = \beta \bullet \widetilde{\beta}(v, w)1, \tag{8.24}$$

$$\phi(e_f)\phi(e_g) = \psi_\alpha(f, g)\ \phi(e_{fg}) \tag{8.25}$$

$$\phi(e_f)\phi(w) = \phi(f \cdot w)\phi(e_f), \tag{8.26}$$

for all $w, v \in W \bullet \widetilde{W}$, $f, g \in U_\alpha$. This is done by a straightforward computation. To prove that the image of ϕ is contained in $L\square_H K$ we must prove that if $w \in W \bullet \widetilde{W}$ then

$$\iota_1(w) \otimes 1 + e_u \otimes \iota_2(w) \in L\square_{H_2} K.$$

This calculation is readily proven. Let us prove that ϕ is a comodule morphism. For the moment we shall denote by $\lambda_{W \bullet \widetilde{W}}$ the coaction of $\mathcal{L}(W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha)$. Let $w = (v_1, w_1) \in W \bullet \widetilde{W}$ then

$$\begin{aligned}(\mathrm{id} \otimes \phi)\lambda_{W \bullet \widetilde{W}}(w) &= (\mathrm{id} \otimes \phi)(v_1 \otimes 1 + w_1(u_1, u_1) \otimes e_{(u_1,u_1)} + (u_1, 1) \otimes w) \\ &= v_1 \otimes 1 \otimes 1 + w_1(u_1, u_1) \otimes e_{(u_1,u_2)} \otimes e_{(u_2,u_1)} \\ &\quad + (u_1, 1) \otimes \iota_1(w) \otimes 1 + (u_1, 1) \otimes e_{(u_1,u_2)} \otimes \iota_2(w).\end{aligned}$$

Let λ denote the coaction of $L\square_H K$ described in (4.4) and

$$\iota_1(w) = (v_1, v_2), \quad \iota_2(w) = (v_2, w_1).$$

We have that

$$\begin{aligned}\lambda(\iota_1(w)\otimes 1) &= (\pi_{H_1}\otimes\pi_{H_1}\otimes\mathrm{id})(v_1\otimes 1\otimes 1 + v_2 u\otimes e_u\otimes 1 + u_1\otimes\iota_1(w)\otimes 1)\\ &= v_1\otimes 1\otimes 1 + u_1\otimes\iota_1(w)\otimes 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(e_u\otimes\iota_2(w)) &= (\pi_{H_1}\otimes\pi_{H_1}\otimes\mathrm{id})\big(uv_2\otimes e_u\otimes 1 + uw_1(u_2,u_1)\otimes e_u\otimes e_{(u_2,u_1)}\\ &\quad + u(u_2,1)\otimes e_u\otimes\iota_2(w)\big)\\ &= (u_1,1)w_1(1,u_1)\otimes e_u\otimes e_{(u_2,u_1)} + (u_1,1)\otimes e_u\otimes\iota_2(w)\\ &= w_1(u_1,u_1)\otimes e_u\otimes e_{(u_2,u_1)} + (u_1,1)\otimes e_u\otimes\iota_2(w).\end{aligned}$$

The last equality follows because u_1 commutes with w_1. Then

$$(\mathrm{id}\otimes\phi)\lambda_{W\bullet\widetilde{W}}(w) = \lambda\phi(w).$$

An easy computation shows that the same equality holds for the group elements in U_α. Clearly the map ϕ is injective. This implies that $\dim(\mathcal{L}(W\bullet\widetilde{W},\beta\bullet\widetilde{\beta},F\bullet\widetilde{F},\psi\bullet\widetilde{\psi}))\leq\dim(L\square_{H_2}K)$, but from Claim 8.4 it follows that both spaces have the same dimension. Therefore ϕ is an isomorphism. □

Let (W,β,F,ψ), $(\widetilde{W},\widetilde{\beta},\widetilde{F},\widetilde{\psi})$ be compatible data with (V,V,u,u,G,G). The spaces $W,\widetilde{W}$ have decompositions $W=W^1\oplus W^2\oplus W^3$, $\widetilde{W}=\widetilde{W}^1\oplus\widetilde{W}^2\oplus\widetilde{W}^3$.

Let $L=\mathcal{K}(W,\beta,F,\psi)$, $K=\mathcal{K}(\widetilde{W},\widetilde{\beta},\widetilde{F},\widetilde{\psi})$. The tensor product $L\otimes_{\Bbbk}K$ has a left H-comodule structure $\delta: L\otimes_{\Bbbk}K\to H_2\otimes_{\Bbbk}L\otimes_{\Bbbk}K$ given by

$$\delta(l\otimes k) = \pi_2(k_{(-1)})\mathcal{S}^{-1}(\pi_2(l_{(-1)}))\otimes l_{(0)}\otimes k_{(0)},$$

for all $l\otimes k\in L\otimes_{\Bbbk}K$. This coaction was already used in (4.5).

Proposition 8.12. *The following assertions hold.*

1. *If ${}_K\mathcal{M}$ is an invertible bimodule category then $\widetilde{W}^2=0$.*
2. *If ${}_K\mathcal{M}$ is an invertible bimodule category then $\widetilde{W}^1=0$.*
3. *If ${}_K\mathcal{M}$ is an invertible bimodule category then*
$$\widetilde{F}=U_\alpha,\quad \widetilde{\psi}=\psi_\alpha,$$
for some $\alpha\in O(G\oplus\widehat{G})$.
4. *If $W^2=W^1=\widetilde{W}^2=\widetilde{W}^1=0$ and $\widetilde{F}=U_{\widetilde{\alpha}}$, $\widetilde{\psi}=\psi_{\widetilde{\alpha}}$, $F=U_\alpha$, $\widetilde{\psi}=\psi_\alpha$ for some $\alpha,\widetilde{\alpha}\in O(G\oplus\widehat{G})$, there is an isomorphism*
$$L\otimes_{\Bbbk}K\simeq N\otimes_{\Bbbk}(L\square_H K)$$
of right $L\square_H K$-modules and left H-comodules, where N is a certain left H-comodule.

Proof. 1. Since ${}_K\mathcal{M}$ is an invertible bimodule category then

$$\big({}_K\mathcal{M}\big)^{\mathrm{op}}\boxtimes_{\mathrm{Rep}(H)}{}_K\mathcal{M}\simeq\mathcal{M}(H,H,K,\overline{K})\simeq\mathrm{Rep}(H_2).$$

For any vector space X and $P\in\mathcal{M}(H,H,K,\overline{K})$ we write $X\otimes_{\Bbbk}P$ the object in the category $\mathcal{M}(H,H,K,\overline{K})$ with structure concentrated in P. Let $\mathcal{N}$ be the full subcategory of $\mathcal{M}(H,H,K,\overline{K})$ consisting of objects P such that $X\overline{\otimes}P\simeq X\otimes_{\Bbbk}P$ for all

$X \in \operatorname{Rep}(H \otimes H^{\mathrm{cop}})$. The category $\mathcal{N}$ is a submodule category of $\mathcal{M}(H, H, K, \overline{K})$. It could not happen that $\mathcal{N}$ equals $\mathcal{M}(H, H, K, \overline{K})$ since $\mathcal{M}(H, H, K, \overline{K})$ is equivalent to $\operatorname{Rep}(H)$. Thus $\mathcal{N}$ must be the null category.

Let us assume that $\widetilde{W}^2 \neq 0$. Let $\langle W^2 \rangle$ be the subalgebra of K generated by elements in W^2. We have inclusions

$$S = \overline{\langle W^2 \rangle} \square_H \langle W^2 \rangle \hookrightarrow \overline{K} \square_{H_2} K \hookrightarrow \overline{K} \otimes_{\Bbbk} K,$$

of left $H \otimes H^{\mathrm{cop}}$-comodule algebras. Note that the coaction of S is trivial, that is, if $\delta : S \to H \otimes H^{\mathrm{cop}} \otimes_{\Bbbk} S$, $\sum k \otimes l \in S$ then $\delta(\sum k \otimes l) = 1 \otimes 1 \otimes \sum k \otimes l$. This implies that for any $X \in \operatorname{Rep}(H \otimes H^{\mathrm{cop}})$ and $M \in {}_S\mathcal{M}$ $X \overline{\otimes} M = X \otimes_{\Bbbk} M$, where the S-action on $X \otimes_{\Bbbk} M$ is concentrated in the second tensorand. From this observation we deduce that for any $M \in {}_S\mathcal{M}$ the object $\overline{K} \otimes_{\Bbbk} K \otimes_S M$ belongs to $\mathcal{N}$. This is a contradiction, which means that $\widetilde{W}^2 = 0$.

2. The assertion follows by using the same argument as in item (1).

3. Let us assume that ${}_L\mathcal{M}$ is the inverse of the bimodule category ${}_K\mathcal{M}$. From the previous results we know that $W^2 = W^1 = \widetilde{W}^2 = \widetilde{W}^1 = 0$. Let us prove that $(L \square_H K)_0 = L_0 \square_{H_0} K_0$. The inclusion $(L \square_H K)_0 \supseteq L_0 \square_{H_0} K_0$ is immediate. Let $\sum l \otimes k \in (L \square_H K)_0$, then

$$\pi_1(l_{(-1)}) \otimes l_{(0)} \in H_0 \otimes_{\Bbbk} L, \qquad \pi_1(k_{(-1)}) \otimes k_{(0)} \in H_0 \otimes_{\Bbbk} K.$$

The only possibility for this to happen is that $l \in L_0, k \in K_0$. Now the result follows from [17, Corollary 5.6].

4. It follows from Proposition 7.9 and from (8.16) that $L \square_H K$ is a twisting C_σ of some coideal subalgebra C of either $\mathcal{A}(V, V, u, u, G, G)$ or $\mathcal{H}(V, V, u, u, G, G)$. This means that there are equivalences of categories

$$^{B}\mathcal{M}_{L \square_H K} \simeq {}^{B^{[\sigma]}}\mathcal{M}_C \simeq {}^{Q}\mathcal{M},$$

where B is either $\mathcal{A}(V, V, u, u, G, G)$ or $\mathcal{H}(V, V, u, u, G, G)$, and $Q = B/BC^+$. The first equivalence is [16, Lemma 2.1] and the second one is standard, see e.g. [20]. Thus any object of ${}^{B}\mathcal{M}_{L \square_H K}$ is equivalent to $B^{[\sigma]} \square_Q N$ for some $N \in {}^{Q}\mathcal{M}$. In particular, since $L \otimes_{\Bbbk} K \in {}^{B}\mathcal{M}_{L \square_H K}$ there exists $N \in {}^{Q}\mathcal{M}$ such that $L \otimes_{\Bbbk} K \simeq B^{[\sigma]} \square_Q N$. Since $B^{[\sigma]} \simeq C \otimes_{\Bbbk} Q$ as right C-modules and left Q-comodules, then $L \otimes_{\Bbbk} K \simeq C_\sigma \otimes_{\Bbbk} N$. □

Theorem 8.13. *If $\alpha, \widetilde{\alpha}$ are elements in $O(G \oplus \widehat{G})$, then there is an equivalence of bimodule categories*

$${}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M} \boxtimes_{\operatorname{Rep}(H)} {}_{\mathcal{L}(\widetilde{W},\widetilde{\beta},\widetilde{\alpha})}\mathcal{M} \simeq {}_{\mathcal{L}(W \bullet \widetilde{W}, \beta \bullet \widetilde{\beta}, \alpha\widetilde{\alpha})}\mathcal{M}.$$

Proof. From Proposition 8.12 (4) we can apply Theorem 4.2 and we get that

$${}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M} \simeq {}_{\mathcal{L}(W,\beta,\mathrm{id}) \square_H \mathcal{L}(V,0,\alpha)}\mathcal{M} \simeq {}_{\mathcal{L}(W,\beta,\mathrm{id})}\mathcal{M} \boxtimes_{\operatorname{Rep}(H)} {}_{\mathcal{L}(V,0,\alpha)}\mathcal{M},$$

where the first isomorphism is (8.16). Then

$${}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M} \boxtimes_{\operatorname{Rep}(H)} {}_{\mathcal{L}(\widetilde{W},\widetilde{\beta},\widetilde{\alpha})}\mathcal{M}$$

is isomorphic to

$$_{\mathcal{L}(W,\beta,\mathrm{id})}\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{\mathcal{L}(V,0,\alpha)}\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{\mathcal{L}(V,0,\widetilde{\alpha})}\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{\mathcal{L}(\widetilde{W},\widetilde{\beta},\mathrm{id})}\mathcal{M}.$$

Using Theorem 8.8 we obtain that this tensor product is isomorphic to

$$_{\mathcal{L}(W,\beta,\mathrm{id})}\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{\mathcal{L}(V,0,\alpha\widetilde{\alpha})}\mathcal{M} \boxtimes_{\mathrm{Rep}(H)} {}_{\mathcal{L}(\widetilde{W},\widetilde{\beta},\mathrm{id})}\mathcal{M},$$

and using again Theorem 4.2 we get the result. □

Define $\mathfrak{B}(V,u,G)$ to be the group of invertible elements in $\mathcal{R}(V,u,G)/\sim$ with product $\bullet$ described in (8.2).

Theorem 8.14. *Let G be a finite group, $u \in G$ be a central element of order 2 and V a finite-dimensional G-module such that $u \cdot v = -v$ for all $v \in V$. There is an isomorphism of groups*

$$\mathrm{BrPic}\,(\mathrm{Rep}(\mathcal{A}(V,u,G))) \simeq \mathfrak{B}(V,u,G).$$

Proof. It follows from Theorem 8.7 that the application

$$\mathcal{R}(V,u,G)^{\times} \to \mathrm{BrPic}\,(\mathrm{Rep}(\mathcal{A}(V,u,G))),\ (W,\beta,\alpha) \mapsto {}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M}$$

is well-defined. It follows from Theorem 7.12 that $(W,\beta,\alpha) \sim (\widetilde{W},\widetilde{\beta},\widetilde{\alpha})$ if and only if the module categories $_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M}$, $_{\mathcal{L}(\widetilde{W},\widetilde{\beta},\widetilde{\alpha})}\mathcal{M}$ are equivalent. Hence we have a well-defined injective map

$$\mathfrak{B}(V,u,G) \to \mathrm{BrPic}\,(\mathrm{Rep}(\mathcal{A}(V,u,G)),\ (W,\beta,\alpha) \mapsto {}_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M}.$$

Proposition 8.11 implies that this map is a group homomorphism. Let us prove that it is surjective. Let $\mathcal{M}$ be an exact invertible $\mathrm{Rep}(\mathcal{A}(V,u,G))$-bimodule category. Then, by Theorem 7.10 there exists a data $(W^1 \oplus W^2 \oplus W^3, \beta, F, \psi)$ compatible with (V,V,u,u,G,G) and an equivalence $\mathcal{M} \simeq {}_{\mathcal{K}(W^1,W^2,W^3,\beta,F,\psi)}\mathcal{M}$ of bimodule categories. By Proposition 8.12 (1) and (2) $W^1 = W^2 = 0$. Also by Proposition 8.12 (3) there exists $\alpha \in O(G \oplus \widehat{G})$ such that $(F,\psi) = (U_\alpha, \psi_\alpha)$, thus $\mathcal{K}(W^1,W^2,W^3,\beta,F,\psi) = \mathcal{L}(W,\beta,\alpha)$.

Since $\mathcal{M}$ is invertible there exists another compatible data $(\widetilde{W},\widetilde{\beta},\widetilde{\alpha})$ such that

$$_{\mathcal{L}(W,\beta,\alpha)}\mathcal{M} \boxtimes_{\mathrm{Rep}(\mathcal{A}(V,u,G))} {}_{\mathcal{L}(\widetilde{W},\widetilde{\beta},\widetilde{\alpha})}\mathcal{M} \tag{8.27}$$

is equivalent to $_{\mathcal{L}(\mathrm{diag}(V),0,\mathrm{diag}(G),1)}\mathcal{M}$. From Theorem 8.13 we conclude that the tensor product category (8.27) is equivalent to the category

$$_{\mathcal{K}(W\bullet\widetilde{W},\beta\bullet\widetilde{\beta},\alpha\widetilde{\alpha})}\mathcal{M}.$$

It follows from Theorem 7.12 that $(W\bullet\widetilde{W},\beta\bullet\widetilde{\beta},\alpha\widetilde{\alpha}) \sim (\mathrm{diag}(V),0,\mathrm{diag}(G),1)$ and therefore $(W,\beta,\alpha) \in \mathfrak{B}(V,u,G)$. This finishes the proof of the Theorem. □

8.4. **Another description of the Brauer-Picard group.** In [9] the authors give a beautiful description of the group $\mathrm{BrPic}(\mathrm{Rep}(\Bbbk G))$ for a finite Abelian group G. This group is isomorphic to the group of automorphism of $G \oplus \widehat{G}$, here $\widehat{G}$ is the group of characters of G, such that they preserve the quadratic form $q : G \oplus \widehat{G} \to \Bbbk$, $q(g, f) = f(g)$. In this section we use the same ideas to give a more compact description of the group $\mathfrak{B}(V, u, G)$.

Let $(W, \beta, \alpha) \in \mathcal{R}(V, u, G)$. Set $\tau(W, \beta)$ the subspace of $V \oplus V^* \oplus V \oplus V^*$ defined by

$$\{(w_1, f_1, w_2, f_2) : (w_1, w_2) \in W,\ (f_1, f_2) \in W^*,\ \widehat{\beta}(w_1, w_2) = f_1 - f_2\}.$$

Recall the definition of $\widehat{\beta}$ given in (2.1). If (W', β', α') is another element in $\mathcal{R}(V, u, G)$ we denote $\tau(W, \beta) \bullet \tau(W', \beta')$ the set of elements (w_1, f_1, w_2, f_2) such that there exists a unique $(v, g) \in V \oplus V^*$ such that $(w_1, f_1, v, g) \in \tau(W, \beta)$ and $(v, g, w_2, f_2) \in \tau(W', \beta')$.

Let $\mathrm{Lag}\,(V, u, G)$ be the set of pairs $(\tau(W, \beta), \alpha)$ where (W, β, α) is an invertible element in $\mathcal{R}(V, u, G)$. If $(\tau(W, \beta), \alpha), (\tau(W', \beta'), \alpha') \in \mathrm{Lag}\,(V, u, G)$ define

$$(\tau(W, \beta), \alpha) \bullet (\tau(W', \beta'), \alpha') = (\tau(W, \beta) \bullet \tau(W', \beta'), \alpha \circ \alpha'). \tag{8.28}$$

Two elements $(\tau(W, \beta), \alpha), (\tau(W', \beta'), \alpha')$ in $\mathrm{Lag}\,(V, u, G)$ are *equivalent* if there exists $(x, y) \in G \times G$ such that

$$(\tau(W', \beta'), \alpha') = (\tau((x, y) \cdot W, (x, y) \cdot \beta), \alpha).$$

We denote by $\overline{\mathrm{Lag}}\,(V, u, G)$ the set of equivalence classes in $\mathrm{Lag}\,(V, u, G)$. The next lemma is a result analogous to [9, Prop. 10.3].

Lemma 8.15. *The set* $\overline{\mathrm{Lag}}\,(V, u, G)$ *is a group with operation defined by* (8.28) *in each equivalence class and identity element the class of* $(\{(v, f, v, f) : v \in V, f \in V^*\}, \mathrm{id})$. *The map* $\tau : \mathfrak{B}(V, u, G) \to \overline{\mathrm{Lag}}\,(V, u, G)$ *that sends the class of* (W, β, α) *to the class of* $(\tau(W, \beta), \alpha)$ *is a group isomorphism.*

Proof. The proof that $\overline{\mathrm{Lag}}\,(V, u, G)$ is a group is straightforward. Let us take (W, β, α), $(W', \beta', \alpha') \in \mathcal{R}(V, u, G)$ and $(w_1, f_1, w_2, f_2) \in \tau(W, \beta) \bullet \tau(W', \beta')$. Then there exists $(v, g) \in V \oplus V^*$ such that $(w_1, f_1, v, g) \in \tau(W, \beta)$ and $(v, g, w_2, f_2) \in \tau(W', \beta')$. Hence

$$\widehat{\beta}(w_1, v) = f_1 - g, \quad \widehat{\beta'}(v, w_2) = g - f_2,$$

which implies that

$$\widehat{\beta \bullet \beta'}(w_1, w_2) = f_1 - f_2.$$

Thus, $(w_1, f_1, w_2, f_2) \in \tau(W \bullet W', \beta \bullet \beta')$ and we have an inclusion $\tau(W, \beta) \bullet \tau(W', \beta') \subseteq \tau(W \bullet W', \beta \bullet \beta')$. The other inclusion is proven similarly. Thus τ is well-defined and injective. By definition of $\mathrm{Lag}\,(V, u, G)$ the map τ is surjective. $\square$

The group $G \times G$ acts on the set of linear maps $T : V \oplus V^* \to V \oplus V^*$ as follows. If $(x, y) \in G \times G$, $(v, f) \in V \oplus V^*$ define

$$((x, y) \cdot T)(v, f) = x^{-1} \cdot T(y \cdot v, y \cdot f). \tag{8.29}$$

The action of G on V^* is given by

$$(x \cdot f)(v) = f(x^{-1} \cdot v),$$

for all $x \in G$, $f \in V^*$, $v \in V$.

Definition 8.16. Let $\mathcal{O}(V, u, G)$ be the set of pairs (T, α) where

(i) $\alpha \in O(G \oplus \widehat{G})$ such that $(u, u) \in U_\alpha$,

(ii) $T : V \oplus V^* \to V \oplus V^*$ is a linear isomorphism such that

$$(x, y) \cdot T = T, \quad \text{for all } (x, y) \in U_\alpha, \tag{8.30}$$

$$T^1(0, f) = 0, \quad T^2(0, f)(T^1(v, 0)) = f(v), \quad \text{for all } f \in V^*, v \in V. \tag{8.31}$$

Here $T(v, f) = (T^1(v, f), T^2(v, f))$ for all $f \in V^*, v \in V$.

Two elements (T, α), (T', α') are *equivalent* if there exists $(x, y) \in G \times G$ such that

$$T' = (x^{-1}, y^{-1}) \cdot T, \quad \alpha = \alpha'.$$

The class of an element $(T, \alpha) \in \mathcal{O}(V, u, G)$ will be denoted by $\overline{(T, \alpha)}$ and the set of equivalence classes will be denoted $\overline{\mathcal{O}}(V, u, G)$.

Remark 8.17. If $(T, \mathrm{id}) \in \mathcal{O}(V, u, G)$ then $T \in \mathrm{Aut}_G(V \oplus V^*)$.

Lemma 8.18. *The set $\overline{\mathcal{O}}(V, u, G)$ is a group with unit element $\overline{(\mathrm{Id}, \mathrm{id})}$ and composition*

$$\overline{(T, \alpha)} \bullet \overline{(T', \alpha')} = \overline{(T \circ T', \alpha \circ \alpha')},$$

for all $\overline{(T, \alpha)}, \overline{(T', \alpha')} \in \overline{\mathcal{O}}(V, u, G)$. □

Theorem 8.19. *There is an isomorphism of groups $\mathfrak{B}(V, u, G) \simeq \overline{\mathcal{O}}(V, u, G)$.*

Proof. Let (T, α) be a representative of a class in $\overline{\mathcal{O}}(V, u, G)$. Define $T^1 : V \oplus V^* \to V, T^2 : V \oplus V^* \to V^*$ by $T(v, f) = (T^1(v, f), T^2(v, f))$ for any $(v, f) \in V \oplus V^*$. Let W_T the subspace of $V \oplus V$ defined as

$$W_T = \{(T^1(v, f), v) : v \in V, f \in V^*\},$$

and the bilinear form $\beta_T : W_T \times W_T \to \Bbbk$ defined by

$$\beta_T((T^1(v_1, f_1), v_1), (T^1(v_2, f_2), v_2)) = T^2(v_1, f_1)(T^1(v_2, f_2)) - f_1(v_2),$$

for all $(v_1, f_1), (v_2, f_2) \in V \oplus V^*$.

Claim 8.5. $(W_T, \beta_T, \alpha) \in \mathcal{R}(V, u, G)$.

Proof of Claim. Let us prove that β_T is U_α-invariant. The other conditions can be easily verified. Let $(g, h) \in U_\alpha$, $(v_1, f_1), (v_2, f_2) \in V \oplus V^*$ then $\beta_T((g, h) \cdot (T^1(v_1, f_1), v_1), (g, h) \cdot (T^1(v_2, f_2), v_2))$ is equal to

$$\begin{aligned}
&= \beta_T((g \cdot T^1(v_1, f_1), h \cdot v_1), (g \cdot T^1(v_2, f_2), h \cdot v_2)) \\
&= \beta_T((T^1(h \cdot v_1, h \cdot f_1), h \cdot v_1), (T^1(h \cdot v_2, h \cdot f_2), h \cdot v_2)) \\
&= T^2(h \cdot v_1, h \cdot f_1)(T^1(h \cdot v_2, h \cdot f_2)) - h \cdot f_1(h \cdot v_2) \\
&= T^2(v_1, f_1)(T^1(v_2, f_2)) - f_1(v_2) \\
&= \beta_T((T^1(v_1, f_1), v_1), (T^1(v_2, f_2), v_2)).
\end{aligned}$$

The second and fourth equalities follow because $(g,h)\cdot T = T$. $\square$

We will establish an isomorphism $\sigma : \overline{\mathcal{O}}(V,u,G) \to \overline{\mathrm{Lag}}\,(V,u,G)$ defined by

$$\sigma\overline{(T,\alpha)} = \overline{(\tau(W_T,\beta_T),\alpha)},$$

for all $\overline{(T,\alpha)} \in \overline{\mathcal{O}}(V,u,G)$. This map does not depend on the representative class of $\overline{(T,\alpha)}$. Let us prove that it is injective. Let $\overline{(T,\alpha)} \in \overline{\mathcal{O}}(V,u,G)$ such that

$$\overline{(\tau(W_T,\beta_T),\alpha)} = \overline{(\{(v,f,v,f) : v\in V, f\in V^*\},\mathrm{id})}.$$

Since $(\{(v,f,v,f) : v\in V, f\in V^*\} = \tau(\mathrm{diag}(V),0)$ there exists an element $(x,y)\in G\times G$ such that

$$U_\alpha = \mathrm{diag}(G), \quad \psi_\alpha = 1, \quad W_T = \{(x\cdot v, y\cdot v) : v\in V\}, \quad \beta_T = 0.$$

This implies that $T^1(v,f) = xy^{-1}\cdot v$ for all $(v,f)\in V\oplus V^*$ and since $\beta_T = 0$ then $T^2(v,f) = xy^{-1}\cdot f$, thus $T = (x,y)^{-1}\cdot \mathrm{Id}$. Hence $\overline{(T,\alpha)} = \overline{(\mathrm{Id},\mathrm{id})}$ and σ is injective. Finally, let us prove that σ is surjective. Let $(\tau(W,\beta),\alpha)\in \mathrm{Lag}\,(V,u,G)$. If $(w_1,f_1,w_2,f_2),(w_1',f_1',w_2,f_2)\in\tau(W,\beta)$ then $(w_1-w_1',0)\in W$ which implies that $w_1 = w_1'$. Also

$$\widehat{\beta}(w_1,w_2) = f_1 - f_2 = f_1' - f_2,$$

thus $f_1' = f_1$. In conclusion the pair (w_1,f_1) depends on (w_2,f_2), therefore there is a linear function $T : V\oplus V^* \to V\oplus V^*$ such that $W = W_T$. If the element $(0,0,v,f)\in W$ then $v=0$, $f=0$, thus T must be injective and consequently bijective. It is not difficult to see that $\beta = \beta_T$. This finishes the proof that σ is surjective and the proof of the Theorem. $\square$

Example 8.20. Suppose $\Bbbk = \mathbb{C}$. Let $\mathbb{Z}_2$ be the cyclic group of order 2 with generator u. Let V be a finite-dimensional vector space such that u acts as -1 on V. Set $H = \wedge(V)\#\Bbbk\mathbb{Z}_2$. Assume $\dim V = 1$, so H is Sweedler's Hopf algebra.

The group $O(\mathbb{Z}_2\oplus\widehat{\mathbb{Z}_2}) = \{\mathrm{id},\gamma\}$, see example 8.3. Note that $U_\gamma = \mathbb{Z}_2\oplus\mathbb{Z}_2$. Define

$$\mathcal{O} = \{A\in SL_2(\mathbb{C}) : A_{12} = 0\}.$$

The Brauer-Picard group of $\mathrm{Rep}(H)$ is isomorphic to the group $\mathcal{O}\times\mathbb{Z}_2$. In particular for any $\xi\in\Bbbk$ the matrices

$$\begin{pmatrix} i & 0 \\ \xi & -i \end{pmatrix}$$

give a one parameter family invertible bimodule categories over $\mathrm{Rep}(H)$ of order 4.

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