ON BRAIDED GROUPOIDS

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Abstract. We study and give examples of braided groupoids, and *a fortiori*, non-degenerate solutions of the quiver-theoretical braid equation.

Introduction

Let V be a vector space over some field and let $R : V \otimes V \to V \otimes V$ be a linear operator. One says that R is a solution of the Quantum Yang-Baxter equation (QYBE, for short) if

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

where as usual $R^{12} = R \otimes \mathrm{id}$, and so on. The study of solutions of the QYBE, motivated by problems in statistical mechanics and low dimension topology, has been a central theme in algebra along the last 25 years. If R is a solution of the QYBE and $\tau : V \otimes V \to V \otimes V$ denotes the usual transposition, then $c := R\tau$ is a solution of the *braid equation*, that is

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \tag{0.1}$$

Thus, there is a bijective correspondence between solutions of the QYBE and solutions of the braid equation.

Drinfeld observed in [D] that both the QYBE and the braid equation have sense if V is just a set and $R : V \times V \to V \times V$ is just a map; again, there is a bijective correspondence between solutions of one and the other. He called this the set-theoretical QYBE and proposed its study as a meaningful problem. Note that any solution of the set-theoretical QYBE gives rise, by linearization, to a solution of the QYBE in the category of vector spaces. An active research on Drinfeld's problem was undertaken by several mathematicians: see [WX], [Hi], [GVB], [ESS], [S], [LYZ1, LYZ2], [GI1, GI2], [GIM] and [O]. See also [EGS], where indecomposable solutions on sets with p elements, p a prime, are classified. Later, Takeuchi gave an alternative presentation of the results by Etingof-Schedler-Soloviev and Lu-Yan-Zhu, with braided groups playing a central rôle. See [T1].

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Now, the braid equation (0.1) has sense in any monoidal category. Another natural monoidal category to consider is the category $\mathrm{Quiv}\,(\mathcal{P})$ of quivers over a fixed set $\mathcal{P}$ with tensor product given by pull-back. The braid equation in $\mathrm{Quiv}\,(\mathcal{P})$ is called the *quiver-theoretical QYBE*, by abuse of notation. A solution of the braid equation in $\mathrm{Quiv}\,(\mathcal{P})$ is called a *braided quiver*. Note that any finite solution of the quiver-theoretical QYBE gives rise, by linearization, to a solution of the QYBE in the category of bimodules over a commutative separable algebra.

The problem of characterizing solutions of the braid equation in $\mathrm{Quiv}\,(\mathcal{P})$ was attacked by Andruskiewitsch, see [A]. In particular Theorem 3.10 in *loc. cit.* shows that there is a bijective correspondence between

- Non-degenerate braided quivers $\mathcal{A}$,
- pairs $(\mathcal{G}, \mathcal{A})$, where $\mathcal{G}$ is a braided groupoid and $\mathcal{A}$ is a representation of $\mathcal{G}$ with certain properties.

In other words, braided groupoids are the fundamental piece of information in the classification of solutions of the quiver-theoretical QYBE. This raises naturally the question of classifying (or at least characterizing) braided groupoids. This is the problem considered in the present paper.

Although braided groupoids appear naturally, by the result quoted above, no systematic investigation of their structure was undertaken up to now. In the paper [AN] a description of matched pair of groupoids in group-theoretical terms is obtained. See also [AM, Thm. 3.1]. The main idea of this work is to use this result to describe braided groupoids in terms of group theory.

This paper is intended to be as self-contained as possible. For this reason we include in section 1.1 some basics definitions concerning groupoids. In section 1.2 we recall the definition of matched pair of groupoid. We explain how to obtain matched pairs of groupoids from a collection (D, V, H, γ), where V, H are subgroups of a finite group D such that V intersects trivially any conjugate of H and $\gamma : V\backslash D/H \to D$ is a section of the canonical projection. To such collection we attach maps $\lambda_V, \lambda_H, \rho_V, \rho_H, \triangleright, \triangleleft$ governing the multiplication of D, with certain cohomological flavor. In section 1.3 we recall the definition of braided groupoid.

Our main result is Theorem 1.11, where we characterize braided groupoids in terms of collections (D, V, H, γ) as before, subject to some restrictions on the maps $\lambda_V, \lambda_H, \rho_V, \rho_H, \triangleright, \triangleleft$. In section 2 we apply the main result to obtain examples under suitable restrictions. Notably, we analyze in subsection 2.1 a class of braided groupoids that we call *handy* and give a complete characterization of them in terms of data including certain "non-associative" group structures. We stress that such structures appear also in some other works in the area [N, B].

In the next subsection, explicit examples of non-handy braided groupoids are also presented. Finally in section 3.1 we compute the braiding for the examples given in section 2.

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1. Braided Groupoids

1.1. **Groupoids.** Recall that a (finite) *groupoid* is a small category (with finitely many arrows), such that every morphism has an inverse. We shall denote a groupoid by $\mathfrak{e}, \mathfrak{s} : \mathcal{G} \rightrightarrows \mathcal{P}$, or simply by $\mathcal{G}$, where $\mathcal{G}$ is the set of arrows, $\mathcal{P}$ is the set of objects and $\mathfrak{e}, \mathfrak{s}$ are the target and source maps.

The set of arrows between two objects P and Q is denoted by $\mathcal{G}(P,Q)$ and we shall also denote $\mathcal{G}(P) := \mathcal{G}(P,P)$. The composition map is denoted by $m : \mathcal{G}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{G}$, and for two composable arrows g and h, that is $\mathfrak{e}(g) = \mathfrak{s}(h)$, the composition will be denoted by juxtaposition: $m(g,h) = gh$.

A *morphism* between two groupoids is a functor of the underlying categories. Two morphisms of groupoids $\phi, \psi : \mathcal{G} \to \mathcal{H}$ are *similar*, denoted $\phi \sim \psi$, if there is a natural transformation between them; that is, if there exists a map $\tau : \mathcal{P} \to \mathcal{H}$ such that

$$\phi(g)\tau(\mathfrak{e}(g)) = \tau(\mathfrak{s}(g))\psi(g), \qquad g \in \mathcal{G}.$$

Two groupoids $\mathcal{G}$, $\mathcal{H}$ are *isomorphic*, and we write $\mathcal{G} \cong \mathcal{H}$, if there are morphisms $\phi : \mathcal{G} \to \mathcal{H}$, $\psi : \mathcal{H} \to \mathcal{G}$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are similar to the corresponding identities.

Any groupoid $\mathcal{G}$ gives rise to a relation on the base $\mathcal{P}$, $P \approx_{\mathcal{G}} Q$ if $\mathcal{G}(P,Q) \neq \emptyset$. A groupoid $\mathfrak{e}, \mathfrak{s} : \mathcal{G} \rightrightarrows \mathcal{P}$ is *connected* if $P \approx_{\mathcal{G}} Q$ for all $P, Q \in \mathcal{P}$.

Let S be an equivalence class in $\mathcal{P}$ and let $\mathcal{G}_S$ denote the corresponding connected groupoid with base S; that is, $\mathcal{G}_S(P,Q) = \mathcal{G}(P,Q)$ for any $P, Q \in S$. Then the groupoid $\mathcal{G}$ is isomorphic to the disjoint union of the connected groupoids $\mathcal{G}_S$: $\mathcal{G} \cong \coprod_{S \in \mathcal{P}/\approx} \mathcal{G}_S$.

If $\mathcal{H}$ and $\mathcal{G}$ are two isomorphic groupoids over the same base $\mathcal{P}$ then there are (non-canonical) isomorphisms $\mathcal{G}(P) \cong \mathcal{H}(P)$ for all $P \in \mathcal{P}$.

A subgroupoid $\mathcal{H}$ of a groupoid $\mathcal{G}$ is *wide* if $\mathcal{H}$ has the same base $\mathcal{P}$ as $\mathcal{G}$.

Let $\mathcal{G} \rightrightarrows \mathcal{P}$ be a groupoid. If $p : \mathcal{E} \to \mathcal{P}$ is a map, a *left action* of $\mathcal{G}$ to $(\mathcal{E}, p)$ is a map $\rightharpoonup : \mathcal{G}\,{}_{\mathfrak{e}}\!\times_p \mathcal{E} \to \mathcal{E}$ such that

$$p(g \rightharpoonup x) = \mathfrak{s}(g), \qquad g \rightharpoonup (h \rightharpoonup x) = gh \rightharpoonup x, \qquad \mathrm{id}_{p(x)} \rightharpoonup x = x, \tag{1.1}$$

for all composable $g, h \in \mathcal{G}$, $x \in \mathcal{E}$. Similarly, a *right action* of $\mathcal{G}$ to $(\mathcal{E}, p)$ is a map $\leftharpoonup : \mathcal{E}\,{}_p\!\times_{\mathfrak{s}} \mathcal{G} \to \mathcal{E}$ such that

$$p(x \leftharpoonup g) = \mathfrak{e}(g), \qquad (x \leftharpoonup g) \leftharpoonup h = x \leftharpoonup gh, \qquad x \leftharpoonup \mathrm{id}_{p(x)} = x, \tag{1.2}$$

for all composable $g, h \in \mathcal{G}$, $x \in \mathcal{E}$

1.2. **Matched Pairs of Groupoids.** We briefly recall some facts about matched pairs of groupoids. See [Ma], [AA] and references therein.

A *matched pair of groupoids* is a collection $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$, where $\mathfrak{e}, \mathfrak{s} : \mathcal{V} \rightrightarrows \mathcal{P}$ and $\mathfrak{e}, \mathfrak{s} : \mathcal{H} \rightrightarrows \mathcal{P}$ are two groupoids over the same base $\mathcal{P}$, $\rightharpoonup: \mathcal{H}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{V} \to \mathcal{V}$ is a left action of $\mathcal{H}$ on $(\mathcal{V}, \mathfrak{s})$, $\leftharpoonup: \mathcal{H}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{V} \to \mathcal{H}$ is a right action of $\mathcal{V}$ on $(\mathcal{H}, \mathfrak{e})$ such that

$$\mathfrak{e}(x \rightharpoonup g) = \mathfrak{s}(x \leftharpoonup g), \quad x \rightharpoonup gh = (x \rightharpoonup g)((x \leftharpoonup g) \rightharpoonup h), \tag{1.3}$$

$$xy \leftharpoonup g = (x \leftharpoonup (y \rightharpoonup g))(y \leftharpoonup g), \tag{1.4}$$

for composable elements $x, y \in \mathcal{H}$ and $g, h \in \mathcal{V}$.

The properties of the actions (1.3) and (1.4) are the classical properties of matched pairs of algebraic objects.

Let $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$ be a matched pair of groupoids. There is an associated *diagonal groupoid* $\mathcal{V} \bowtie \mathcal{H}$ with set of arrows $\mathcal{V}\,{}_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{H}$, base $\mathcal{P}$, source, target, composition and identity given by

$$\begin{aligned} &\mathfrak{s}(g, x) = \mathfrak{s}(g), \quad \mathfrak{e}(g, x) = \mathfrak{e}(x), \\ &(g, x)(h, y) = (g(x \rightharpoonup h), (x \leftharpoonup h)y), \quad \mathrm{id}_P = (\mathrm{id}_P, \mathrm{id}_P), \end{aligned}$$

$g, h \in \mathcal{V}$, $x, y \in \mathcal{H}$, $P \in \mathcal{P}$. The groupoids $\mathcal{V}$ and $\mathcal{H}$ can be seen as wide subgroupoids of $\mathcal{V} \bowtie \mathcal{H}$. Then we have an exact factorization of groupoids $\mathcal{V} \bowtie \mathcal{H} = \mathcal{V}\mathcal{H}$, that is; for every $z \in \mathcal{V} \bowtie \mathcal{H}$ there are unique elements $x \in \mathcal{V}, g \in \mathcal{H}$ such that $z = xg$. Conversely, if $\mathcal{D} = \mathcal{V}\mathcal{H}$ is an exact factorization of groupoids then there are actions $\leftharpoonup$, $\rightharpoonup$ such that $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$ form a matched pair of groupoids, and $\mathcal{D} \simeq \mathcal{V} \bowtie \mathcal{H}$.

Let us fix a connected groupoid $\mathcal{D} \rightrightarrows \mathcal{P}$ and a point $O \in \mathcal{P}$. Set $D = \mathcal{D}(O)$. For each $P \in \mathcal{P}$ we fix $\tau_P \in \mathcal{D}(O, P)$.

In the following we shall study exact factorizations $\mathcal{D} = \mathcal{V}\mathcal{H}$ where $\mathcal{V}$ and $\mathcal{H}$ are connected wide subgroupoids. In this case we can assume that $\tau_P \in \mathcal{V}(O, P)$. There is no harm to assume that $\tau_O = 1$. We shall denote $V = \mathcal{V}(O), H = \mathcal{H}(O)$.

The following lemma will be useful to describe examples of braided groupoids in group-theoretical terms.

Lemma 1.1. *Under the above considerations there is a bijection between the following data.*

i) *Exact factorizations $\mathcal{D} = \mathcal{V}\mathcal{H}$, where $\mathcal{V}, \mathcal{H}$ are connected wide subgroupoids of $\mathcal{D}$,*
ii) *matched pair of groupoids $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$ with $\mathcal{V}$, $\mathcal{H}$ connected, such that $\mathcal{D} \cong \mathcal{V} \bowtie \mathcal{H}$ and*

iii) *collections* (V, H, γ) *where* G, H *are subgroups of* D, $\gamma : \mathcal{P} \to D$ *is a (necessarily) injective map, and the following conditions are fulfilled*

$$D = \coprod_{P \in \mathcal{P}} V \gamma_P H, \tag{1.5}$$

$$V \bigcap z H z^{-1} = \{1\} \tag{1.6}$$

for every $z \in D$.

We shall say that the collection (D, V, H, γ) satisfying conditions of Lemma 1.1 (iii) is associated to the matched pair $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$ or, equivalently, to the exact factorization $\mathcal{D} = \mathcal{V}\mathcal{H}$.

Proof. For the implications (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii) see [AM, Thm. 3.1].

Assume now that V, H are subgroups of D and $\gamma : \mathcal{P} \to D$ is a map such that equations (1.5), (1.6) are fulfilled. Define the wide subgroupoids $\mathcal{V}$ and $\mathcal{H}$ by

$$\mathcal{H}(P, Q) := \tau_P^{-1} \gamma_P H \gamma_Q^{-1} \tau_Q, \quad \mathcal{V}(P, Q) := \tau_P^{-1} V \tau_Q.$$

By construction $\mathcal{D} = \mathcal{V}\mathcal{H}$ is an exact factorization. □

Remark 1.2. We can always assume that $\gamma_O = 1$.

Remark 1.3. Observe that under conditions of Lemma 1.1 (iii) there is a bijection $\mathcal{P} \cong V \backslash D / H$ and via this identification the map γ is a section of the canonical projection. Conditions (1.5), (1.6) imply that $\mid D \mid = \mid V \mid \mid H \mid \#\mathcal{P}$.

Summarizing, to obtain an exact factorization of connected groupoids we need a group D, two subgroups V and H of D such that V intersects trivially all conjugates of H. Take $\mathcal{P}$ the set of double cosets $V \backslash D / H$ and $\gamma : \mathcal{P} \to D$ is any section of the canonical projection. Some examples of such collections are the following:

- V, H subgroups of D with coprime orders,
- $D = VC$ an exact factorization of groups and H is a subgroup of C.

The following basic observation will be used repeated times.

Lemma 1.4. *Assume that* (D, V, H, γ) *is a collection satisfying the conditions of Lemma 1.1 (iii), then for any* $z \in D$ *there exists* $g \in V, x \in H$ *and* $P \in \mathcal{P}$ *uniquely determined such that* $z = g \gamma_P x$.

Proof. The existence is clear. Assume that $g' \gamma_Q x' = g \gamma_P x$, then $P = Q$ and $g^{-1} g' = \gamma_P x x'^{-1} \gamma_P^{-1} \in V \bigcap \gamma_P H \gamma_P^{-1}$, hence $g = g'$ and $x = x'$. □

Assume that (D, V, H, γ) is associated to the matched pair $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$. Thanks to Lemma 1.4 we shall introduce a family of maps. In the next section these maps will be used to write conditions for a groupoid to be braided. Concretely, the maps are

$$\triangleright : H \times V \to V, \ \triangleleft : H \times V \to H,$$

$$(\, ; \,) : H \times V \to \mathcal{P},$$

such that

$$xg = (x \triangleright g)\gamma_{(x;g)}(x \triangleleft g), \tag{1.7}$$

for all $x \in H, g \in V$. Let us also define maps

$$\lambda_V : \mathcal{P} \times V \times \mathcal{P} \to V, \ \ \rho_V : \mathcal{P} \times V \times \mathcal{P} \to H,$$
$$(\, ; \ ; \,) : \mathcal{P} \times V \times \mathcal{P} \to \mathcal{P}$$

and maps

$$\lambda_H : \mathcal{P} \times H \times \mathcal{P} \to V, \ \ \rho_H : \mathcal{P} \times H \times \mathcal{P} \to H,$$
$$< ; \ ; >: \mathcal{P} \times H \times \mathcal{P} \to \mathcal{P}$$

such that

$$\gamma_P g \gamma_Q = \lambda_V(P, g, Q)\gamma_{(P;g;Q)}\rho_V(P, g, Q), \tag{1.8}$$

$$\gamma_P x \gamma_Q = \lambda_H(P, x, Q)\gamma_{<P;x;Q>}\rho_H(P, x, Q), \tag{1.9}$$

for all $P, Q \in \mathcal{P}$, $g \in V, x \in H$.

In the next section we shall study exact factorizations $\mathcal{D} = \mathcal{V}\mathcal{H}$ with $\mathcal{V} \cong \mathcal{H}$. In that case the groups V, H are isomorphic.

If (D, V, H, γ) is associated to the exact factorization $\mathcal{D} = \mathcal{V}\mathcal{H}$, and $\phi : H \to V$ is an isomorphism we shall also denote by ϕ the isomorphism $\phi : \mathcal{H} \to \mathcal{V}$ given by

$$\phi(\tau_P^{-1}\gamma_P g \gamma_Q^{-1}\tau_Q) = \tau_P^{-1}\phi(g)\tau_Q.$$

Given such an isomorphism ϕ, we define the map $m : \mathcal{D} \to \mathcal{V}$ as the composition

$$\mathcal{D} \xrightarrow{\simeq} \mathcal{V} \bowtie \mathcal{H} \xrightarrow{\mathrm{id}\times\phi} \mathcal{V} \,_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{V} \xrightarrow{\mu} \mathcal{V}, \tag{1.10}$$

where $\mu : \mathcal{V} \,_{\mathfrak{e}}\!\times_{\mathfrak{s}} \mathcal{V} \longrightarrow \mathcal{V}$ is the composition.

Using Lemma 1.1 the map $m : \mathcal{D} \to \mathcal{V}$ can be explicitly written as follows.

Lemma 1.5. *Let $\alpha \in \mathcal{D}(P, Q)$, if $\alpha = \tau_P^{-1} g \gamma_R x \tau_Q$ for some $g \in V, x \in H, R \in \mathcal{P}$ then*

$$m(\alpha) = \tau_P^{-1} g \lambda_H(R, x, Q)\phi(\rho_H(R, x, Q))\tau_Q.$$

As a particular case if $\alpha \in \mathcal{D}(O, O)$, $\alpha = g\gamma_R x$ then $m(\alpha) = g\phi(x)$.

Proof. If we have a decomposition $\alpha = \beta_1\beta_2$ where $\beta_1 \in \mathcal{V}, \beta_2 \in \mathcal{H}$ then, by definition, $m(\alpha) = \beta_1\phi(\beta_2)$. Note that if $\alpha = \tau_P^{-1} g \gamma_R x \tau_Q$ then

$$\begin{aligned}\alpha &= \tau_P^{-1} g \gamma_R x \gamma_Q \gamma_Q^{-1}\tau_Q = \tau_P^{-1} g \lambda_H(R, x, Q)\gamma_{<R;x;Q>}\rho_H(R, x, Q)\gamma_Q^{-1}\tau_Q \\ &= \tau_P^{-1} g \lambda_H(R, x, Q)\tau_{<R;x;Q>}\tau^{-1}_{<R;x;Q>}\gamma_{<R;x;Q>}\rho_H(R, x, Q)\gamma_Q^{-1}\tau_Q\end{aligned}$$

where

$$\tau_P^{-1} g \rho_H(R,x,Q)\tau_{<R;x;Q>} \in \mathcal{V}(P, <R;x;Q>),$$
$$\tau^{-1}_{<R;x;Q>}\gamma_{<R;x;Q>}\lambda_H(R,x,Q)\gamma_Q^{-1}\tau_Q \in \mathcal{H}(<R;x;Q>, Q).$$

Therefore

$$\begin{aligned} m(\alpha) &= \tau_P^{-1} g \lambda_H(R,x,Q)\tau_{<R;x;Q>}\phi(\tau^{-1}_{<R;x;Q>}\gamma_{<R;x;Q>}\rho_H(R,x,Q)\gamma_Q^{-1}\tau_Q) \\ &= \tau_P^{-1} g \lambda_H(R,x,Q)\phi(\rho_H(R,x,Q))\tau_Q. \end{aligned}$$

Since for all $R \in \mathcal{P}$, $x \in H$ $\lambda_H(R,x,O) = 1$, and $\rho_H(R,x,O) = x$ the second assertion follows. □

1.3. **Braided Groupoids.** The notion of *braided groupoid* was introduced in [A] in order to study the quiver-theoretical Yang-Baxter equation.

Definition 1.6 ([A])**.** A *braided groupoid* is a collection $(\mathcal{V}, \rightharpoonup, \leftharpoonup)$ where $\mathcal{V} \rightrightarrows \mathcal{P}$ is a groupoid, $(\mathcal{V}, \mathcal{V}, \rightharpoonup, \leftharpoonup)$ is a matched pair of groupoids and for every pair $(f,g) \in \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V}$ the following equation holds:

$$fg = (f \rightharpoonup g)(f \leftharpoonup g). \tag{1.11}$$

If $(\mathcal{V}, \rightharpoonup, \leftharpoonup)$ is a braided groupoid then the map $c : \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V} \to \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V}$ defined by

$$c(\alpha, \beta) = (\alpha \rightharpoonup \beta, \alpha \leftharpoonup \beta) \tag{1.12}$$

satisfies the braid equation.

Let $(\mathcal{V}, \mathcal{H}, \rightharpoonup, \leftharpoonup)$ be a matched of groupoids, and $\phi : \mathcal{H} \to \mathcal{V}$ a groupoid isomorphism, recall the diagonal groupoid $\mathcal{D}$ and the map $m : \mathcal{D} \to \mathcal{V}$ as in the previous section.

Associated to this matched pair of groupoids there is a new pair of actions (that we denote with the same symbol) $\rightharpoonup, \leftharpoonup : \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V} \to \mathcal{V}$, and they are defined by

$$g \rightharpoonup h := \phi^{-1}(g) \rightharpoonup h, \quad g \leftharpoonup h := \phi\left(\phi^{-1}(g) \leftharpoonup h\right),$$

for all composable $g, h \in G$. Since ϕ is a groupoid morphism, the collection $(\mathcal{V}, \mathcal{V}, \rightharpoonup, \leftharpoonup)$ is a matched pair of groupoids.

Lemma 1.7. *The following statements are equivalent.*

i) $(\mathcal{V}, \rightharpoonup, \leftharpoonup)$ *is a braided groupoid,*
ii) *the map* $m : \mathcal{D} \to \mathcal{V}$ *is a groupoid morphism.*

Proof. Let $\mu : \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V} \longrightarrow \mathcal{V}$ be the composition. Since $m = \zeta(\mathrm{id}_{\mathcal{V}} \otimes \phi)\mu$, where $\zeta : \mathcal{D} \xrightarrow{\cong} \mathcal{V} \bowtie \mathcal{H}$, and $(\mathrm{id}_{\mathcal{V}} \otimes \phi)$ is a groupoid morphism, then m is a groupoid morphism if and only if μ is a groupoid morphism. Then the proof follows from [A, Lemma 2.9], where it is proven that $(\mathcal{V}, \rightharpoonup, \leftharpoonup)$ is braided if and only if the composition map μ is a groupoid morphism. □

Without lose of generality we can assume that the groupoid $\mathcal{V}$ is connected. If $\mathcal{V}$ is not connected then $\mathcal{V}$ is similar to the disjoint union of connected groupoids

$$\mathcal{V} \cong \coprod_{S\in\mathcal{P}/\approx} \mathcal{V}_S.$$

Lemma 1.8. *With the notation above $\mathcal{V}$ is braided if and only if for any $S \in \mathcal{P}/\approx$ $\mathcal{V}_S$ is a braided groupoid.*

Proof. The sufficiency is clear. Assume that $\mathcal{V}$ is braided. We only need to show that, for any $S \in \mathcal{P}/\approx$, $\mathcal{V}_S$ is stable under the actions $\rightharpoonup, \leftharpoonup$.

Let $f, g \in \mathcal{V}_S$. Using (1.1), (1.2) we know that

$$\mathfrak{s}(f \rightharpoonup g) = \mathfrak{s}(f), \quad \mathfrak{e}(f \leftharpoonup g) = \mathfrak{e}(g).$$

Since $\mathfrak{s}(f), \mathfrak{e}(g) \in S$ then $f \rightharpoonup g, f \leftharpoonup g \in \mathcal{V}_S$. □

Definition 1.9. We shall say that (D, V, H, γ) is a *braided groupoid datum* if the associated connected groupoid $\mathcal{V}$ is braided, or, equivalently if the map $m : \mathcal{D} \to \mathcal{V}$ is a groupoid morphism.

Remark 1.10. The matched pair $(\mathcal{V}, \mathcal{V}, \rightharpoonup, \leftharpoonup)$ and the map $m : \mathcal{D} \to \mathcal{V}$ both depend on the choice of the isomorphism ϕ. Sometimes the isomorphism ϕ will be clear from the context. We shall denote (D, V, H, γ, ϕ) when special emphasis is needed.

The next result gives necessary and sufficient conditions on the collection (D, V, H, γ, ϕ) to be a braided groupoid datum.

Theorem 1.11. *The collection (D, V, H, γ, ϕ) is a braided groupoid datum if and only if*

$$g = \lambda_V(P, g, Q)\ \phi(\rho_V(P, g, Q)), \tag{1.13}$$

$$\phi(x) = \lambda_H(P, x, Q)\ \phi(\rho_H(P, x, Q)), \tag{1.14}$$

$$\phi(x)g = (x \triangleright g)\ \phi(x \triangleleft g), \tag{1.15}$$

for all $P, Q \in \mathcal{P}$, $g \in V$, $x \in H$.

Proof. Assume that (D, V, H, γ) is a braided groupoid datum. Set $\alpha = \gamma_P g \gamma_Q$ $= \lambda_V(P, g, Q)\ \gamma_{(P;g;Q)}\rho_V(P, g, Q)$, then using Lema 1.5 we have that $m(\alpha) = \lambda_V(P, g, Q)\ \phi(\rho_V(P, g, Q))$. Since m is a groupoid morphism then $m(\alpha) = m(\gamma_P)m(g)\ m(\gamma_Q) = g$, hence we have proved equation (1.13). Equations (1.14), (1.15) follows in a similar way using equations (1.7), (1.9).

Suppose that equations (1.13), (1.14), (1.15) are fulfilled. Let α, $\beta \in \mathcal{D}$ two composable elements, then $\alpha = \tau_P^{-1} g \gamma_R x \tau_Q$, $\beta = \tau_Q^{-1} h \gamma_S y \tau_M$ for some $g, h \in V$, $x, y \in H$ and $P, Q, M, R, S \in \mathcal{P}$. We shall prove that $m(\alpha\beta) = m(\alpha)m(\beta)$.

Lemma 1.5 together with equation (1.14) implies that

$$m(\alpha)=\tau_P^{-1}g\phi(x)\tau_Q, \quad m(\beta)=\tau_Q^{-1}h\phi(y)\tau_M.$$

Let us compute $\alpha\beta$. Define the elements $X,Y\in\mathcal{P}$ by $X:=(R;(x\triangleright h);(x;h))$, $Y:=<X;\rho_V(R,(x\triangleright h),(x;h))(x\triangleleft h);S>$, then

$$\begin{aligned}\alpha\beta &= \tau_P^{-1}g\gamma_R xh\gamma_S y\tau_M=\tau_P^{-1}g\gamma_R(x\triangleright h)\gamma_{(x;h)}(x\triangleleft h)\gamma_S y\tau_M\\ &= \tau_P^{-1}g\lambda_V(R,(x\triangleright h),(x;h))\gamma_X\rho_V(R,(x\triangleright h),(x;h))(x\triangleleft h)\gamma_S y\tau_M\\ &= \tau_P^{-1}g\lambda_V(R,(x\triangleright h),(x;h))\lambda_H(X,\rho_V(R,(x\triangleright h),(x;h))(x\triangleleft h),S)\gamma_Y\\ &\quad \rho_H(X,\rho_V(R,(x\triangleright h),(x;h))(x\triangleleft h),S)y\tau_M.\end{aligned}$$

Therefore

$$\begin{aligned}m(\alpha\beta) &= \tau_P^{-1}g\lambda_V(R,(x\triangleright h),(x;h))\lambda_H(X,\rho_V(R,(x\triangleright h),(x;h))(x\triangleleft h),S)\\ &\qquad \phi\left(\rho_H(X,\rho_V(R,(x\triangleright h),(x;h))(x\triangleleft h),S)\right)\phi\left(y\right)\tau_M\\ &= \tau_P^{-1}g\rho_V(R,(x\triangleright h),(x;h))\phi\left(\rho_V(R,(x\triangleright h),(x;h))\right)\phi\left(x\triangleleft h\right)\phi\left(y\right)\tau_M\\ &= \tau_P^{-1}g(x\triangleright h)\phi\left(x\triangleleft h\right)\phi\left(y\right)\tau_M\\ &= \tau_P^{-1}g\phi\left(x\right)h\phi\left(y\right)\tau_M=m(\alpha)m(\beta).\end{aligned}$$

The second equality by (1.14), the third by (1.13) and the fourth by (1.15).
□

2. Examples

In this section we shall give examples of braided groupoid data.

2.1. **Handy braided groupoids.**

In this section we study braided groupoid datum with the following properties:

$$\bullet\ VH=HV, \tag{2.1}$$

$$\bullet\ \gamma(\mathcal{P})H=H\gamma(\mathcal{P}), \text{ and} \tag{2.2}$$

$$\bullet\ \gamma(\mathcal{P})V=V\gamma(\mathcal{P}). \tag{2.3}$$

This class of braided groupoids is the simplest to deal with. A braided groupoid $\mathcal{V}$ whose associated braided groupoid datum (D,V,H,γ) satisfies equations (2.1), (2.2), (2.3) will be called *handy braided groupoid.*

Let F be a group, and $\triangleright,\triangleleft$: $F\times F\to F$ a left (respect. right) action on the set F. Let $\mathcal{P}$ be a set together with an operation $\mathcal{P}\times\mathcal{P}\to\mathcal{P}$, $(P,Q)\mapsto PQ$, not necessarily associative, such that

(i) There exists $O\in\mathcal{P}$ satisfying $PO=OP=P$, for all $P\in\mathcal{P}$,
(ii) for any $P\in\mathcal{P}$ there is a unique $Q\in\mathcal{P}$ such that $PQ=QP=O$. This element will be denoted by P^{-1}.

Let $\rightharpoondown : F \times \mathcal{P} \to \mathcal{P}$ be a group action and $\sigma : \mathcal{P} \times \mathcal{P} \to F$ a map such that

$$\sigma(P, O) = \sigma(O, P) = 1, \tag{2.4}$$

$$g \rightharpoondown O = O, \tag{2.5}$$

$$\sigma(P, P^{-1}) = 1, \tag{2.6}$$

for all $g \in F,\ P \in \mathcal{P}$.

Denote by $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ the set $F \times \mathcal{P} \times F$ with multiplication given by

$$(g, P, x)(h, Q, y) := (g(x \triangleright h)\sigma(X, Y), XY, \sigma(X, Y)^{-1}(x \triangleleft h)y),$$

for every $g, h, x, y \in F, P, Q \in \mathcal{P}$, where

$$X = (x \triangleright h)^{-1} \rightharpoondown P, \quad Y = (x \triangleleft h) \rightharpoondown Q.$$

Equations (2.4), (2.5) implies that $(1, O, 1)$ is a unit for this product.

Under certain compatibilities of the maps $\sigma, \triangleright, \triangleleft, \rightharpoondown$ this multiplication makes $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ into a group. This is the next lemma.

Lemma 2.1. *Keep the notation above. The set $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ is a group with unit $(1, O, 1)$ if and only if the following conditions are fulfilled.*

$$(F, F, \triangleright, \triangleleft) \quad \textit{is a matched pair of groups,} \tag{2.7}$$

$$(PQ)(\sigma(P, Q)^{-1} \rightharpoondown R) = (\sigma(Q, R)^{-1} \rightharpoondown P)(QR), \tag{2.8}$$

$$\sigma(Q, R)\, \sigma\big(\sigma(Q, R)^{-1} \rightharpoondown P, QR\big) = \sigma(P, Q)\, \sigma\big(PQ, \sigma(P, Q)^{-1} \rightharpoondown R\big), \tag{2.9}$$

$$(g \rightharpoondown P)(g \rightharpoondown Q) = (g \triangleleft \sigma(P, Q)) \rightharpoondown PQ, \tag{2.10}$$

$$g \triangleright \sigma(P, Q) = \sigma(g \rightharpoondown P, g \rightharpoondown Q), \tag{2.11}$$

$$(g \triangleright \sigma(P, Q))\, (g \triangleleft \sigma(P, Q)) = g\, \sigma(P, Q), \tag{2.12}$$

for all $g \in F,\ P, Q, R \in \mathcal{P}$

Proof. Assume that $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ is a group. From equalities

$$(1, O, x)\, ((1, O, y)(g, O, 1)) = ((1, O, x)(1, O, y))\, (g, O, 1),$$
$$(1, O, x)\, ((g, O, 1)(h, O, 1)) = ((1, O, x)(g, O, 1))\, (h, O, 1),$$

follow that $(F, F, \triangleright, \triangleleft)$ is a matched pair of groups. Equations (2.8), (2.9) follow from the equation

$$(1, P, 1)\, ((1, Q, 1)(1, R, 1)) = ((1, P, 1)(1, Q, 1))\, (1, R, 1).$$

Equations (2.10), (2.11), (2.12) can be deduced from the equality

$$(1, O, g)\, ((1, P, 1)(1, Q, 1)) = ((1, O, g)(1, P, 1))\, (1, Q, 1).$$

Assume that equations (2.7) to (2.12) are fulfilled. First we shall prove that the product in $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ is associative. We claim that it is enough to prove that

$$((g,P,1)(1,O,x))\,(h,q,y) = (g,P,1)\,((1,O,x)(h,q,y))\,, \tag{2.13}$$

$$((1,O,x)(h,Q,y))\,(f,R,z) = (1,O,x)\,((h,Q,y)(f,R,z))\,, \tag{2.14}$$

$$((g,P,1)(h,Q,y))\,(f,R,z) = (g,P,1)\,((h,Q,y)(f,R,z))\,, \tag{2.15}$$

for al $P,Q,R \in \mathcal{P}$, $x,y,z,h,f,g \in F$. Indeed, let $P,Q,R \in \mathcal{P}$, $x,y,z,h,f,g \in F$ then

$$\begin{aligned}(g,P,x)\,((h,Q,y)(f,R,z)) &= ((g,P,1)(1,O,x))\,((h,Q,y)(f,R,z))\\ &= (g,P,1)\,((1,O,x)((h,Q,y)(f,R,z)))\\ &= (g,P,1)\,(((1,O,x)(h,Q,y))(f,R,z))\\ &= ((g,P,1)((1,O,x)(h,Q,y)))\,(f,R,z)\\ &= (((g,P,1)(1,O,x))(h,Q,y))\,(f,R,z)\\ &= ((g,P,x)(h,Q,y))\,(f,R,z).\end{aligned}$$

The second equality by (2.13), the third by (2.14), the fourth by (2.15) and the fifth again by (2.13).

Equation (2.13) follows by a direct calculation. Equation (2.14) follows from (2.7), (2.11) and (2.12). Equation (2.15) follows from (2.8),(2.9) (2.10) and (2.11). The inverse of an element is

$$(g,P,x)^{-1} = (x^{-1} \triangleright g^{-1}, (x^{-1} \triangleleft g^{-1})g \rightharpoondown P^{-1}), x^{-1} \triangleleft g^{-1}).$$

□

When the map σ or the action $\rightharpoondown$ are trivial, conditions in Lemma 2.1 are easy to handle, as the following corollaries show.

Corollary 2.2. *Assume that $(F, \triangleright, \triangleleft)$ is a matched pair of groups, $\mathcal{P}$ is a group with identity O, and $\sigma : \mathcal{P} \times \mathcal{P} \to F$ is a map such that*

- $\sigma(P,O) = \sigma(O,P) = 1,$
- $\sigma(P,P^{-1}) = 1$
- $\sigma(Q,R)\sigma(P,QR) = \sigma(P,Q)\sigma(PQ,R),$

for all $P,Q,R \in \mathcal{P}$. If in addition we have that

- $g \triangleright \sigma(P,Q) = \sigma(P,Q), \quad g \triangleleft \sigma(P,Q) = \sigma(P,Q)^{-1} g \sigma(P,Q),$

for all $g \in V$, $P,Q \in \mathcal{P}$, then $F_\sigma \bowtie \mathcal{P}_\sigma \bowtie F$ is a group, where $\rightharpoondown$ is trivial. □

Corollary 2.3. *Assume that $(F, \triangleright, \triangleleft)$ is a matched pair of groups, $\mathcal{P}$ is a group with identity O and $\rightharpoondown$ is a left action of F on $\mathcal{P}$ by group automorphisms. Then $F \bowtie \mathcal{P} \bowtie F$ is a group, here the map σ is assumed to be trivial.* □

Let us assume that $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ is a group, or, equivalently, the properties (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) hold.

Define the subgroups V, H of $F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$ by $V := F \times O \times 1$, $H := 1 \times O \times F$. The map $\gamma : \mathcal{P} \to F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F$, is the inclusion; $\gamma(P) = (1, P, 1)$.

Then the collection $(F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F, V, H, \gamma)$ satisfies conditions of Lemma 1.1 (iii).

Theorem 2.4. *If $(F, \triangleright, \triangleleft)$ is a braided group, then $(F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F, V, H, \gamma)$ is a braided groupoid datum and the associated braided groupoid is handy.*

Reciprocally if (D, V, H, γ, ϕ) is a braided groupoid datum and the associated braided groupoid is handy, then $(V, \triangleright, \triangleleft)$ is a braided group, $\mathcal{P}$ has an operation that satisfies (i), (ii), there are maps $\sigma : \mathcal{P} \times \mathcal{P} \to V$, $\rightharpoondown : V \times \mathcal{P} \to \mathcal{P}$ such that $D \cong V \bowtie_\sigma \mathcal{P}_\sigma \bowtie V$ and γ is the inclusion via this isomorphism.

Proof. If $h, y \in F$, $P, Q \in \mathcal{P}$ then

$$\begin{aligned}
\lambda_V(P, (h, 1, 1), Q) &= (h\sigma(h^{-1} \rightharpoondown P, Q), 1, 1),\\
\rho_V(P, (h, 1, 1), Q) &= (1, 1, \sigma(h^{-1} \rightharpoondown P, Q)^{-1}),\\
\lambda_H(P, (1, 1, y), Q) &= (\sigma(P, y \rightharpoondown Q), O, 1),\\
\rho_H(P, (1, 1, y), Q) &= (1, O, \sigma(P, y \rightharpoondown Q)^{-1} y),\\
(1, 1, y) \triangleright (h, 1, 1) &= (y \triangleright h, 1, 1),\\
(1, 1, y) \triangleleft (h, 1, 1) &= (1, 1, y \triangleleft h).
\end{aligned}$$

Therefore the first assertion follows from Theorem 1.11.

Let (D, V, H, γ) be a braided groupoid datum such that equations (2.1), (2.2), (2.3) are satisfied. Abusing of the notation we define $\triangleright, \triangleleft : V \times V \to V$ by

$$g \triangleright h := \phi^{-1}(g) \triangleright h, \quad g \triangleleft h := \phi\left(\phi^{-1}(g) \triangleleft h\right),$$

for all $g, h \in V$. Since $VH = HV$ then $(x; g) = O$ for all $x \in H, g \in V$. Associativity axiom of the group D implies that $(V, V, \triangleright, \triangleleft)$ is a matched pair of groups. Equation (1.15) implies that $(V, \triangleright, \triangleleft)$ is a braided group.

Define the following composition $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$, $PQ := (P; 1; Q)$. Clearly O is a unit for this operation. The existence of inverse in D translates in the existence of the inverse in $\mathcal{P}$.

Define the maps $\sigma : \mathcal{P} \times \mathcal{P} \to V$, $\rightharpoondown : V \times \mathcal{P} \to \mathcal{P}$ by

$$\sigma(P, Q) := \lambda_V(P, 1, Q), \quad g\,\gamma_P := \gamma_{g \rightharpoondown P}\, g'$$

for all $P, Q \in \mathcal{P}$, $g \in V$, where g' is some element in G that depends on g and P. Since the map m is a groupoid morphism then $m(g\,\gamma_P) = m(g) = g$, and therefore $g = g'$. Hence the map $\rightharpoondown$ is defined by the equation

$$g\,\gamma_P := \gamma_{g \rightharpoondown P}\, g.$$

Equation (1.13) implies that $\rho_V(P, 1, Q) = \phi^{-1}(\sigma(P, Q)^{-1})$.

Define $f : D \to V \bowtie_\sigma \mathcal{P}_\sigma \bowtie V$ by

$$f(g\gamma_P x) = (g, P, \phi(x)),$$

for all $g \in V, P \in \mathcal{P}, x \in H$. This is a well defined group isomorphism. This ends the proof of the theorem. □

In particular, Theorem 2.4, in presence of Corollaries 2.2, 2.3, shows that there is a way to produce many examples of braided groupoid datum. For example, take $(F, \triangleright, \triangleleft)$ any braided group, $\mathcal{P}$ a group such that F acts on $\mathcal{P}$ by group automorphism; or take $F, \mathcal{P}$ two groups with a normalized 2-cocycle $\sigma : F \times F \to \mathcal{P}$, $\triangleright : F \times F \to F$ the trivial action and $\triangleleft : F \times F \to F$ the adjoint action.

Corollary 2.5. *Let (D, V, H, γ) be a braided groupoid datum, where V and H are normal subgroups of D. Then the associated braided groupoid $\mathcal{V}$ is handy, moreover the action $\rightharpoondown$ is trivial.*

Proof. Since V is normal $\gamma_P g \gamma_P^{-1} \in V$, for all $P \in \mathcal{P}$, $g \in V$. Hence, $\gamma_P g = g' \gamma_P$ for some $g' \in V$. Since (D, V, H, γ) is a braided groupoid datum then $g = g'$. Analogously we prove that $\gamma_P x = x\gamma_P$ and $gx = xg$ for all $x \in H$, $g \in V$, $P \in \mathcal{P}$. □

2.2. **Non-handy examples.** Let $(A, \triangleright, \triangleleft)$ be a matched pair of groups. Let $\mathcal{P}$ be a group, and let $\psi : A \times A \to \mathcal{Z}(\mathcal{P})$, $\mathcal{Z}(\mathcal{P})$ the center of $\mathcal{P}$, be a map such that for any $a, b, c \in A$

$$\psi(a, bc) = \psi(a, b)\psi(a \triangleleft b, c), \tag{2.16}$$

$$\psi(ab, c) = \psi(a, b \triangleright c)\psi(b, c). \tag{2.17}$$

Define the group D whose underlying set is $A \times \mathcal{P} \times A$ and multiplication given by

$$(a, P, c)(x, Q, z) = (a(c \triangleright x), P\psi(c, x)Q, (c \triangleleft x)z),$$

for any $a, c, x, z \in A$, $P, Q \in \mathcal{P}$. A straightforward computation shows that this operation is associative.

Let $V = A \times 1 \times 1$, $H = 1 \times 1 \times A$ and $\gamma : \mathcal{P} \to D$, $\gamma_P = (1, P, 1)$,

Lemma 2.6. *If $(A, \triangleright, \triangleleft)$ is a braided group then the collection (D, V, H, γ) is a braided groupoid datum.*

Proof. For any $P, Q \in \mathcal{P}$, $a, b \in A$ we have that

$$\lambda_V(P, (a, 1, 1), Q) = (a, 1, 1), \quad \rho_V(P, (a, 1, 1), Q) = 1$$
$$\lambda_H(P, (1, 1, a), Q) = 1, \quad \rho_H(P, (1, 1, a), Q) = (1, 1, a),$$
$$(1, 1, a) \triangleright (b, 1, 1) = (a \triangleright b, 1, 1), \quad (1, 1, a) \triangleleft (b, 1, 1) = (1, 1, a \triangleleft b).$$

Then the Lemma follows by applying Theorem 1.11. □

If $a, z \in A$ then

$$(a,1,1)(1,1,z) = (a,1,z), \quad (1,1,z)(a,1,1) = (z \triangleright a, \psi(z,a), z \triangleleft a).$$

Thus, $VH = HV$ if and only if $\psi = 1$.

Remark 2.7. There are many collections $(A, \triangleright, \triangleleft, \psi)$, where $(A, \triangleright, \triangleleft)$ is a braided group and ψ is a map satisfying (2.16), (2.17). For example take A any group, $\triangleright$ the adjoint action, $\triangleleft$ the trivial action and ψ any bicharacter, that is $\psi : A \times A \to \mathcal{Z}(\mathcal{P})$ such that

$$\psi(a, bc) = \psi(a,b)\psi(a,c),$$
$$\psi(ab, c) = \psi(a,c)\psi(b,c),$$

for all $a, c, x, z \in A$.

This class of examples arise from the following general observation. Let (D, V, H, γ, ϕ) be any braided groupoid datum. Recall the map $(;) : V \times H \to \mathcal{P}$ defined by equation (1.7). If we assume that for all $P \in \mathcal{P}, g \in G, x \in H$

$$\gamma_P g = g\gamma_P, \quad \gamma_P x = x\gamma_P,$$

then the map $\psi : V \times V \to \mathcal{P}$ defined by

$$\psi(g,h) = (g; \phi^{-1}(h)),$$

for all $g, h \in V$, satisfies equations (2.16) and (2.17). Consider the following operation in $\mathcal{P}$; $P.Q = (P;1;Q)$. Since $xg\gamma_P = \gamma_P xg$ for all $x \in H, g \in V, P \in \mathcal{P}$ then

$$(x \triangleright g)\gamma_{(x;g)}\gamma_P(x \triangleleft g) = (x \triangleright g)\gamma_P\gamma_{(x;g)}(x \triangleleft g),$$

and thus, $(x; g) \in \mathcal{Z}(\mathcal{P})$ for all $x \in H, g \in V$.

3. The Braiding

In this section we explicitly compute the braiding for the braided data given in the previous section.

Let (D, V, H, γ, ϕ) be a braided groupoid datum and let $\mathcal{D} = \mathcal{V}\mathcal{H}$ be the associated exact factorization of groupoids. Let $\alpha \in \mathcal{H}, \beta \in \mathcal{V}$ then

$$\alpha = \tau_P^{-1}\,\gamma_P x \gamma_Q^{-1}\,\tau_Q, \quad \beta = \tau_Q^{-1}\,g\,\tau_R,$$

for some $P, Q, R \in \mathcal{P}, g \in V, x \in H$. Then

$$\alpha\beta = \tau_P^{-1}\,\gamma_P x \gamma_Q^{-1} g\,\tau_R.$$

Since $\alpha\beta = (\alpha \rightharpoonup \beta)(\alpha \leftharpoonup \beta)$, the determination of the actions $\rightharpoonup, \leftharpoonup$ relies on the explicit calculation of $\gamma_P x \gamma_Q^{-1} g$. This will be done in the following for the examples explained above.

3.1. **The braiding for handy braided groupoids.** Let $\mathcal{V}$ be a handy braided groupoid and $(F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F, V, H, \gamma)$ its braided groupoid datum. Let also $\mathcal{D} = \mathcal{V}\mathcal{H}$ be the exact factorization associated to the collection $(F \bowtie_\sigma \mathcal{P}_\sigma \bowtie F, V, H, \gamma)$.

Lemma 3.1. *IF $P, Q, R \in \mathcal{P}, x, y \in F$ then*

$$\begin{aligned}\gamma_P(1, O, x)\gamma_Q^{-1}(y, O, 1)\gamma_R =&(\sigma(P, Q^{-1})(\sigma(P, Q^{-1})^{-1}x \triangleright y)\sigma(S, T), O, 1)\gamma_{ST}\\ &(1, O, \sigma(P, Q^{-1})^{-1}x \triangleleft y),\end{aligned}$$

where

$$S = (\sigma(P, Q^{-1})^{-1} \triangleright y)^{-1} \rightharpoondown (PQ^{-1}), \tag{3.1}$$

$$T = (\sigma(P, Q^{-1})^{-1} \triangleright y) \rightharpoondown R. \tag{3.2}$$

Proof. Straightforward. □

Let $(\alpha, \beta) \in \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V}$. Then there exists $P, Q, R \in \mathcal{P}$, $x, y \in F$ such that

$$\alpha = \tau_P^{-1}(x, O, 1)\tau_Q, \quad \beta = \tau_Q^{-1}(y, O, 1)\tau_R.$$

Then

$$\begin{aligned}\phi^{-1}(\alpha)\beta &= \tau_P^{-1}\gamma_P(1, O, x)\gamma_Q^{-1}(y, O, 1)\tau_R\\ &= \tau_P^{-1}\gamma_P(1, O, x)\gamma_Q^{-1}(y, O, 1)\gamma_R\gamma_R^{-1}\tau_R\\ &= \tau_P^{-1}(\sigma(P, Q^{-1})(\sigma(P, Q^{-1})^{-1}x \triangleright y)\sigma(S, T), O, 1)\tau_{ST}\\ &\quad \tau_{ST}^{-1}\gamma_{ST}(1, O, \sigma(P, Q^{-1})^{-1}x \triangleleft y)\iota_R^{-1}\tau_R,\end{aligned}$$

Where $S, T \in \mathcal{P}$ are as in Lemma 3.1. Since

$$\tau_P^{-1}\,(\sigma(P, Q^{-1})(\sigma(P, Q^{-1})^{-1}x \triangleright y)\sigma(S, T), O, 1)\,\tau_{ST} \in \mathcal{V}(P, ST),$$

$$\tau_{ST}^{-1}\,\gamma_{ST}(1, O, \sigma(P, Q^{-1})^{-1}x \triangleleft y)\gamma_R^{-1}\,\tau_R \in \mathcal{H}(ST, R),$$

then

$$\begin{aligned}\alpha \rightharpoonup \beta &= \tau_P^{-1}\,(\sigma(P, Q^{-1})(\sigma(P, Q^{-1})^{-1}x \triangleright y)\sigma(S, T), O, 1)\,\tau_{ST}\\ \alpha \leftharpoonup \beta &= \tau_{ST}^{-1}\,(1, O, \sigma(P, Q^{-1})^{-1}x \triangleleft y)\,\tau_R\end{aligned}$$

As a consequence of these calculations we have the following result.

Proposition 3.2. *The braiding for the handy braided groupoid $\mathcal{V}$ is given by the formula*

$$\begin{aligned}c(\alpha, \beta) =&(\tau_P^{-1}\,(\sigma(P, Q^{-1})(\sigma(P, Q^{-1})^{-1}x \triangleright y)\sigma(S, T), O, 1)\,\tau_{ST},\\ &\tau_{ST}^{-1}\,(1, O, \sigma(P, Q^{-1})^{-1}x \triangleleft y)\,\tau_R)\end{aligned}$$

where $\alpha = \tau_P^{-1}(x, O, 1)\tau_Q, \beta = \tau_Q^{-1}(y, O, 1)\tau_R$ and S, T are given by equations (3.1), (3.2) □.

Remark 3.3. When $\#\mathcal{P} = 1$ then formula in Proposition 3.2 is $c(x, y) = (x \triangleright y, x \triangleleft y)$, which is the braid formula for the braided group $(F, \triangleright, \triangleleft)$.

3.2. **The braiding for the examples in subsection 2.2.** Let $(A, \triangleright, \triangleleft)$ be a braided group, $\mathcal{P}$ be a group. Let also $\psi : A \times A \to \mathcal{Z}(\mathcal{P})$ be a map satisfying (2.16), (2.17). Let (D, V, H, γ) be the braided groupoid datum as in example 2.2. Let $\mathcal{D} = \mathcal{V}\mathcal{H}$ be the exact factorization associated to (D, V, H, γ).

Lemma 3.4. *Let $a, b \in A$, $P, Q, R \in \mathcal{P}$ then*

$$\gamma_P(1, 1, a)\gamma_Q^{-1}(b, 1, 1)\gamma_R = (a \triangleright b, \psi(a, b)PQ^{-1}R, a \triangleleft b). \quad \square$$

Let $(\alpha, \beta) \in \mathcal{V}_{\mathfrak{e}} \times_{\mathfrak{s}} \mathcal{V}$. Then there exists $P, Q, R \in \mathcal{P}$, $a, b \in A$ such that

$$\alpha = \tau_P^{-1}(a, 1, 1)\tau_Q, \quad \beta = \tau_Q^{-1}(b, 1, 1)\tau_R,$$

Then

$$\begin{aligned}
\phi^{-1}(\alpha)\beta &= \tau_P^{-1}\gamma_P(1, 1, a)\gamma_Q^{-1}(b, 1, 1)\tau_R \\
&= \tau_P^{-1}\gamma_P(1, 1, a)\gamma_Q^{-1}(b, 1, 1)\gamma_R\gamma_R^{-1}\tau_R \\
&= \tau_P^{-1}\,(a \triangleright b, \psi(a, b)PQ^{-1}R, a \triangleleft b)\gamma_R^{-1}\,\tau_R \\
&= \tau_P^{-1}\,(a \triangleright b, 1, 1)\,\tau_S\ \tau_S^{-1}\gamma_S(1, 1, a \triangleleft b)\gamma_R^{-1}\,\tau_R,
\end{aligned}$$

where $S = \psi(a, b)PQ^{-1}R \in \mathcal{P}$. Since

$$\tau_P^{-1}\,(a \triangleright b, 1, 1)\,\tau_S \in \mathcal{V}(P, S),$$

$$\tau_S^{-1}\gamma_S(1, 1, a \triangleleft b)\gamma_R^{-1}\,\tau_R \in \mathcal{H}(S, R),$$

then

$$\begin{aligned}
\alpha \rightharpoonup \beta &= \tau_P^{-1}\,(a \triangleright b, 1, 1)\,\tau_S \\
\alpha \leftharpoonup \beta &= \tau_S^{-1}(1, 1, a \triangleleft b)\,\tau_R.
\end{aligned}$$

Proposition 3.5. *If $\alpha = \tau_P^{-1}(a, 1, 1)\tau_Q$, $\beta = \tau_Q^{-1}(b, 1, 1)\tau_R$ then the braiding for the examples 2.2 are given by the formula*

$$c(\alpha, \beta) = (\tau_P^{-1}\,(a \triangleright b, 1, 1)\,\tau_S, \tau_S^{-1}(1, 1, a \triangleleft b)\,\tau_R),$$

where $S = \psi(a, b)PQ^{-1}R$. $\square$

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