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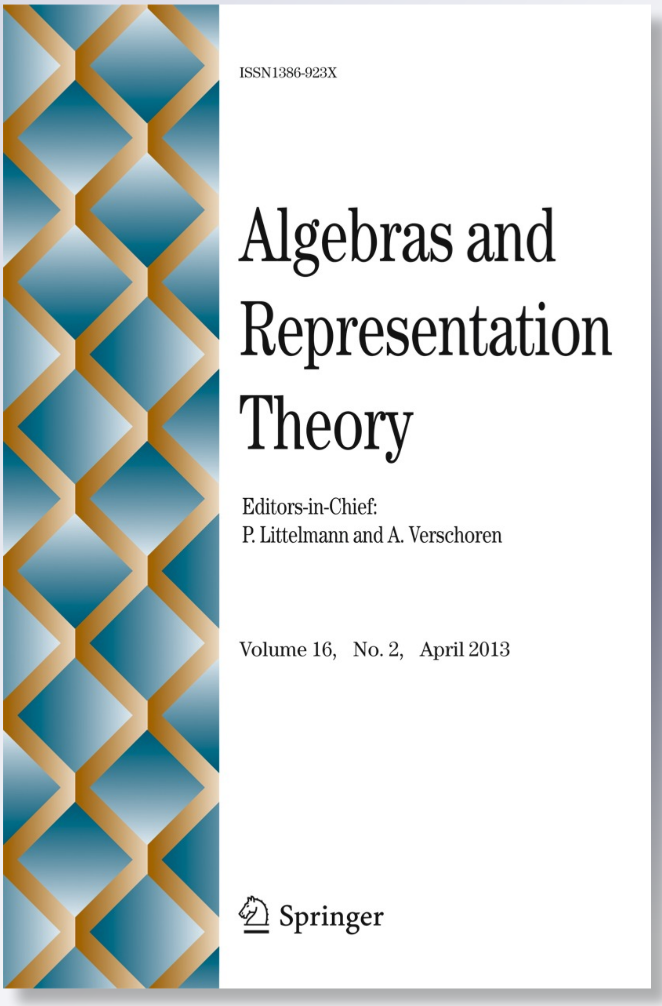

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Families of Finite-Dimensional Hopf Algebras with the Chevalley Property

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Abstract We introduce and study new families of finite-dimensional Hopf algebras with the Chevalley property that are not pointed nor semisimple arising as twistings of quantum linear spaces. These Hopf algebras generalize the examples introduced in Andruskiewitsch et al. (Mich Math J 49(2):277–298, 2001), Etingof and Gelaki (Int Math Res Not 14:757–768, 2002, Math Res Lett 8:249–255, 2001).

Keywords Hopf algebras · Nichols algebras

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1 Introduction

A *Drinfeld's twist*, or simply a *twist*, for a Hopf algebra H is an invertible element $J \in H \otimes H$ satisfying a certain non-linear equation, which in some sense is dual to the notion of a 2-cocycle. Any twist gives rise to a new Hopf algebra H^J constructed over the same underlying algebra H such that if R is a quasitriangular structure for H then $R^J = J_{21} R J^{-1}$ is a quasitriangular structure for H^J. The twisting procedure is a very powerful tool to construct new examples of (quasitriangular) Hopf algebras from old and well-known ones.

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It has been used in [9, 12] to construct simple Hopf algebras by twisting group algebras, see also [10]. In [1] the authors introduce some families of triangular Hopf algebras by twisting finite supergroups. The idea of the construction is roughly the following. They begin with a finite group G with a central element of order two $u \in G$, V a finite-dimensional representation of G where u acts as -1 and $B \in S^2(V)$. The tensor product of the exterior algebra $\wedge(V)$ and the group algebra $\Bbbk G$ is a braided Hopf algebra in the category of Yetter–Drinfeld modules over $\mathbb{Z}_2$. The twisting of $\wedge(V)\otimes\Bbbk G$ by the exponential e^B of the symmetric element B gives a new braided Hopf algebra and by bosonization a (usual) Hopf algebra.

This idea was developed further in a series of papers [6–8] culminating in the classification of finite-dimensional triangular Hopf algebras. Also in [5] further properties of this family have been studied.

The main goal of this paper is to generalize this construction replacing the group $\mathbb{Z}_2$ by an arbitrary finite Abelian group Γ and V by a quantum linear space over G such that Γ is in the center of G. As expected, the role of the exterior algebra is played by the corresponding Nichols algebra $\mathfrak{B}(V)$. This construction gives in some cases new examples of finite-dimensional Hopf algebras.

The contents of the paper are the following. In Section 3 we briefly recall the definition of quantum linear space over a finite group and the definition of twist over a braided Hopf algebra. We associate to any twist over a braided Hopf algebra a twist over the corresponding bosonization. For any quantum linear space V we construct families of twists in $\mathfrak{B}(V)$ using the quantum version of the exponential map.

In Section 4 for any quantum linear space V over a finite group G and a subspace $W \subseteq V$, a subgroup $F \subseteq G$ and a twist J for the group algebra $\Bbbk F$ we introduce the definition of the Hopf algebras $A(V, G, W, F, J, \mathcal{D})$ and we study the particular class of these Hopf algebras when $V = W$, $F = \Gamma$, $J = 1\otimes 1$, that we denote by $A(V, G, \mathcal{D})$. In the non-trivial cases these finite-dimensional Hopf algebras are new. We describe some isomorphisms of the Hopf algebras $A(V, G, \mathcal{D})$ and we study the algebra structure of $A(V, G, \mathcal{D})^*$ from which we give necessary and sufficient conditions for the Hopf algebra $A(V, G, \mathcal{D})$ to be pointed.

2 Preliminaries and Notation

Throughout the paper $\Bbbk$ will denote an algebraically closed field of characteristic 0 and all vector spaces and algebras are assumed to be over $\Bbbk$.

If G is a group and (V, δ) is a left G-comodule we shall denote $V_g = \{v \in V : \delta(v) = g\otimes v\}$ for all $g \in G$.

If $q \in \Bbbk$, $n \in \mathbb{N}$ denote $(n)_q = 1 + q + \cdots + q^{n-1}$ and as usual the q-factorial numbers: $n!_q = (1)_q(2)_q \ldots (n)_q$. The quantum Gaussian coefficients are defined for $0 \leq k \leq n$ by

$$\binom{n}{k}_q = \frac{n!_q}{(n-k)!_q\, k!_q}.$$

The following technical result will be needed later.

Lemma 2.1 *Let $a, i, j, N \in \mathbb{N}$, $q \in \Bbbk$ such that $q^N = 1$, $1 < N$. If $0 \le a, i, j < N$ and $i + j = N + a$, then*

$$\sum_{k=0}^{a} q^{k(k-j)} \binom{j}{k}_q \binom{i}{a-k}_q = 1 \tag{2.1}$$

Proof Let x, y be elements in an algebra such that $yx = q\,xy$. Equation 2.1 follows by using the quantum binomial formula to expand $(x+y)^a(x+y)^N$ and $(x+y)^i(x+y)^j$, and then compute the corresponding coefficient of the term $x^a y^N$. □

If x is an element in an algebra such that $x^N = 0$ and $q \in \Bbbk$, the *q-exponential map* [11] is defined by

$$\exp_q(x) = \sum_{n=0}^{N-1} \frac{1}{(n)!_q}\; x^n.$$

The element $\exp_q(x)$ is invertible, see [11, Prop. IV.2.6], and the inverse is given by

$$\exp_q(x)^{-1} = \sum_{n=0}^{N-1} \frac{(-1)^n\, q^{n(n-1)/2}}{(n)!_q}\; x^n.$$

We shall need the following result.

Lemma 2.2 [13, Lemma 3.2] *If q is a N-th root of 1, $xy = q\,yx$ and $x^i y^{N-i} = 0$ for any $i = 0 \dots N$ then* $\exp_q(x+y) = \exp_q(x)\exp_q(y)$.

3 Twists in Quantum Linear Spaces

We shall describe some families of twists over quantum linear spaces and describe twists for some (usual) Hopf algebras constructed from braided Hopf algebras.

3.1 Quantum Linear Spaces

We shall recall the definition of quantum linear spaces introduced in [2]. Let G be a finite group and $\theta \in \mathbb{N}$. Let $g_1, \dots, g_\theta \in G$, $\chi_1, \dots, \chi_\theta \in \widehat{G}$. Denote $q_{ij} = \chi_j(g_i)$, $q_i = q_{ii}$, for any $i, j = 1, \dots, \theta$. Let N_i be the order of q_i which is assumed to be finite and $N_i > 1$. The collection $(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ is a *datum for a quantum linear space* if

$$g_i h = h g_i, \quad \chi_i\chi_j = \chi_j\chi_i \quad \text{for all } i, j, \text{ and all } h \in G, \tag{3.1}$$

$$q_{ij}q_{ji} = 1 \quad \text{for all } i \neq j. \tag{3.2}$$

We shall denote by Γ the Abelian group generated by $\{g_i : i = 1, \dots, \theta\}$. Let V be the vector space with basis $\{x_1, \dots, x_\theta\}$. With the following maps V is an object in ${}^G_G\mathcal{YD}$:

$$\delta(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)\, x_i.$$

We shall denote $V = V(g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$. The associated Nichols algebra [3] $\mathfrak{B}(V)$ is the algebra generated by elements $\{x_1, \ldots, x_\theta\}$ subject to relations

$$x_i^{N_i} = 0, \quad x_i x_j = q_{ij}\, x_j x_i \text{ if } i \neq j. \tag{3.3}$$

$\mathfrak{B}(V)$ is a braided Hopf algebra in ${}^G_G\mathcal{YD}$ with coproduct determined by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ for all $i = 1, \ldots, \theta$. The braided Hopf algebra $\mathfrak{B}(V)$ is called a *quantum linear space*.

Using the quantum binomial formula we get that for any i, n

$$\Delta(x_i^n) = \sum_{k=0}^{n} \binom{n}{k}_{q_i} x_i^k \otimes x_i^{n-k}.$$

There is an isomorphism $\mathfrak{B}(V)^* \simeq \mathfrak{B}(V^*)$ of braided Hopf algebras. For any $i = 1, \ldots, \theta$ and $0 \leq r_i < N_i$ define $X_1^{r_1} \ldots X_\theta^{r_\theta}$ the element in $\mathfrak{B}(V)^*$ determined by

$$\langle X_1^{r_1} \ldots X_\theta^{r_\theta}, x_1^{s_1} \ldots x_\theta^{s_\theta} \rangle = \begin{cases} \prod_{i=1}^{\theta} (r_i)!_{q_i} & \text{if } r_i = s_i, \text{ for all } i = 1, \ldots, \theta \\ 0 & \text{otherwise.} \end{cases}$$

The braided Hopf algebra $\mathfrak{B}(V)^*$ is generated by elements $X_1, \ldots, X_\theta$ subject to relations (3.3). The coproduct is determined by $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$ for all $i = 1, \ldots, \theta$.

3.2 Twists in Braided Hopf Algebras

Let $\mathcal{H}$ be a braided Hopf algebra in the category ${}^G_G\mathcal{YD}$. A *twist* for $\mathcal{H}$ is an invertible element $J \in \mathcal{H} \otimes \mathcal{H}$ such that $\delta(J) = 1 \otimes J$ and

$$(\Delta \otimes \mathrm{id})(J)(J \otimes 1) = (\mathrm{id} \otimes \Delta)(J)(1 \otimes J), \quad (\varepsilon \otimes \mathrm{id})(J) = 1 = (\mathrm{id} \otimes \varepsilon)(J). \tag{3.4}$$

Here $\delta : \mathcal{H} \otimes \mathcal{H} \to \Bbbk G \otimes \mathcal{H} \otimes \mathcal{H}$ is the coaction of $\mathcal{H} \otimes \mathcal{H}$ in the category ${}^G_G\mathcal{YD}$. As for usual Hopf algebras there is a new braided Hopf algebra structure on the vector space $\mathcal{H}$ with the same algebra structure and coproduct given by $\Delta^J(h) = J^{-1}\Delta(h)J$, for all $h \in \mathcal{H}$. This new braided Hopf algebra is denoted by $\mathcal{H}^J$. See [1].

Two twists $J, \widetilde{J} \in \mathcal{H}$ are *gauge equivalent* if there exists an invertible element $c \in \mathcal{H}$ such that $\epsilon(c) = 1$, $\delta(c) = 1 \otimes c$, $g \cdot c = c$ for all $g \in G$ and

$$\widetilde{J} = \Delta(c) J (c^{-1} \otimes c^{-1}).$$

In this case the map $\phi : \mathcal{H}^{\widetilde{J}} \to \mathcal{H}^J$, $\phi(h) = chc^{-1}$ is an isomorphism of braided Hopf algebras.

Remark 3.1 The product of elements in $\mathcal{H} \otimes \mathcal{H}$ or in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, as in Eq. 3.4, is the product in the tensor product algebra as an object in the corresponding braided tensor category.

The following technical Lemma will be of great use later, the proof is straightforward.

Lemma 3.2 *Let J, J' be twists for $\mathcal{H}$. Assume that*

$$(1\otimes J)(\mathrm{id}\otimes\Delta)(J') = (\mathrm{id}\otimes\Delta)(J')(1\otimes J), \tag{3.5}$$

$$(J\otimes 1)(\Delta\otimes\mathrm{id})(J') = (\Delta\otimes\mathrm{id})(J')(J\otimes 1). \tag{3.6}$$

Then the product JJ' is a twist for $\mathcal{H}$.

For any element $J \in \mathcal{H}\otimes\mathcal{H}$ we shall denote by $\mathcal{H}^*_{(J)}$ the vector space $\mathcal{H}^*$ with product given by

$$\langle X * Y, h\rangle = \langle X, h_{(1)}(h_{(2)})_{(-1)} \cdot J^1\rangle\langle Y, (h_{(2)})_{(0)} J^2\rangle, \tag{3.7}$$

for all $X, Y \in \mathcal{H}, h \in \mathcal{H}$.

Lemma 3.3 *The above product in $\mathcal{H}^*_{(J)}$ is associative if and only if J satisfies*

$$(\Delta\otimes\mathrm{id})(J)(J\otimes 1) = (\mathrm{id}\otimes\Delta)(J)(1\otimes J).$$

In particular $\mathcal{H}^*_{(J)}$ is an algebra with unit ε in the category ${}^G_G\mathcal{YD}$ if and only if J is a twist for $\mathcal{H}$.

Let us mention some applications of Lemma 3.7. Denote W the 1-dimensional vector space generated by x. The space W is a Yetter–Drinfeld module over G with structure maps given by

$$\delta(x) = g\otimes x, \qquad f \cdot x = \chi(f)\, x,$$

where $g \in G$, $\chi \in \widehat{G}$, and $q = \chi(g)$. Assume that q has order $N > 1$ and that $g^N = 1$. Thus $\mathfrak{B}(W) = \Bbbk[x]/(x^N)$. For any $\xi \in \Bbbk$ denote

$$J_\xi = 1\otimes 1 + \sum_{k=1}^{N-1} \frac{\xi}{(N-k)!_q k!_q}\; x^{N-k}\otimes x^k. \tag{3.8}$$

Proposition 3.4 *J_ξ is a twist for $\mathfrak{B}(W)$.*

Proof Clearly J_ξ is invertible with inverse given by $J_{-\xi}$, also $(\varepsilon\otimes\mathrm{id})(J_\xi) = 1 = (\mathrm{id}\otimes\varepsilon)(J_\xi)$. Let us prove that $\mathfrak{B}(W)^*_{(J_\xi)}$ with the product (3.7) is associative. The vector space $\mathfrak{B}(W)^*$ has a basis consisting of elements $\{X^i : i = 0, \dots, N-1\}$, where $\langle X^i, x^j\rangle = \delta_{i,j}\,(i!)_q$ for any $i, j \in \{1, \dots, N-1\}$. For any $0 \le i, j < N$ we have that

$$X^i * X^j = \begin{cases} X^{i+j} & \text{if } i+j < N \\ \xi X^a & \text{if } i+j = N+a, 0 \le a < N. \end{cases} \tag{3.9}$$

Hence $X^i * \left(X^k * X^j\right) = \left(X^i * X^k\right) * X^j$ for any i, j, k and the product $*$ is associative. Let us prove that $X^i * X^j = \xi\, X^a$ when $i+j = N+a$, the other case is straightforward. By definition if $h \in \mathfrak{B}(W)$ then $\langle X^i * X^j, h\rangle$ is equal to

$$\langle X^i, h_{(1)}\rangle\langle X^j, h_{(2)}\rangle + \sum_{k=1}^{N-1} \frac{\xi}{(N-k)!_q k!_q}\; \langle X^i, h_{(1)}h_{(2)(-1)} \cdot x^{N-k}\rangle\langle X^j, h_{(2)(0)}x^k\rangle.$$

Thus it is clear that if $h \neq x^a$ then $\langle X^i * X^j, h\rangle = 0$. We have also that $\langle X^i, (x^a)_{(1)}\rangle\langle X^j, (x^a)_{(2)}\rangle = 0$, then $\langle X^i * X^j, x^a\rangle$ is equal to

$$= \sum_{k=1}^{N-1}\sum_{l=0}^{a} \binom{a}{l}_q \frac{\xi}{(N-k)!_q k!_q} \langle X^i, x^{a-l} g^l \cdot x^k\rangle\langle X^j, x^l x^{N-k}\rangle$$

$$= \sum_{k=1}^{N-1}\sum_{l=0}^{a} \binom{a}{l}_q \frac{\xi\, q^{lk}}{(N-k)!_q k!_q} \langle X^i, x^{a-l+k}\rangle\langle X^j, x^{N-k+l}\rangle$$

$$= \sum_{l=0}^{a} \binom{a}{l}_q \frac{\xi\;\, q^{l(i+l-a)}\;\, i!_q\;\, j!_q}{(N-i-l+a)!_q (i+l-a)!_q}$$

$$= \xi\, a!_q \sum_{l=0}^{a} q^{l(l-j)} \binom{j}{l}_q \binom{i}{a-l}_q = \xi\, a!_q.$$

The last equality follows from Eq. 2.1. □

Let $V = V(g_1, g_2)$ be the 2-dimensional Yetter–Drinfeld module for some datum of a quantum linear space. Assume that $g_1 g_2 = 1$ and that $N_1 = N = N_2$, $q_1 = q = q_2^{-1}$ is a N-th primitive root of unity. Then $\mathfrak{B}(V)$ is the algebra generated by x, y subject to relations

$$x^N = 0 = y^N, \quad xy = q\, yx.$$

For any $a \in \Bbbk$ set $B = a\, x \otimes y$ and $J_a = \exp_q(B)$. It follows by a straightforward computation that $(1\otimes B)(B\otimes 1) = (B\otimes 1)(1\otimes B)$, hence the exponentials commute: $\exp_q(1\otimes B)\exp_q(B\otimes 1) = \exp_q(B\otimes 1)\exp_q(1\otimes B)$.

Proposition 3.5 *J_a is a twist for $\mathfrak{B}(V)$ in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.*

Proof The proof goes in a similar way as the proof of [13, Thm. 3.3]. It is immediate to verify that $(\varepsilon\otimes \mathrm{id})(J_a) = 1 = (\mathrm{id}\otimes\varepsilon)(J_a)$. First note that

$$(\Delta\otimes \mathrm{id})(\exp_q(B)) = \exp_q((\Delta\otimes \mathrm{id})(B)), \quad \exp_q(B)\otimes 1 = \exp_q(B\otimes 1)$$

and

$$(\mathrm{id}\otimes\Delta)(\exp_q(B)) = \exp_q((\mathrm{id}\otimes\Delta)(B)), \quad 1\otimes \exp_q(B) = \exp_q(1\otimes B).$$

Let us denote $C = a(x\otimes 1\otimes y)$, then

$$(\Delta\otimes \mathrm{id})(B) = C + 1\otimes B, \quad (\mathrm{id}\otimes\Delta)(B) = C + B\otimes 1.$$

It follows easily that $q\,(B\otimes 1)C = C(B\otimes 1)$, $(1\otimes B)C = q\,C(1\otimes B)$ and for any $k = 0\dots N$ we have that $C^k(B\otimes 1)^{N-k} = C^k(1\otimes B)^{N-k} = 0$. Then using Lemma 2.2 we get that

$$\begin{aligned}(\Delta\otimes\mathrm{id})(J_a)(J_a\otimes 1) &= \exp_q((\Delta\otimes\mathrm{id})(B))\exp_q(B\otimes 1)\\ &= \exp_q(C + (1\otimes B))\exp_q(B\otimes 1)\\ &= \exp_q(C)\exp_q(1\otimes B)\exp_q(B\otimes 1)\\ &= \exp_q(C)\exp_q(B\otimes 1)\exp_q(1\otimes B)\\ &= \exp_q(C + B\otimes 1)\exp_q(1\otimes B)\\ &= \exp_q((\mathrm{id}\otimes\Delta)(B))\exp_q(1\otimes B) = (\mathrm{id}\otimes\Delta)(J_a)(1\otimes J_a).\end{aligned}$$

□

3.3 A Hopf Algebra Associated to a Braided Hopf Algebra

Let Γ be a finite Abelian group and $\mathcal{H} \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that Γ is a subgroup of the group-like elements in $\mathcal{H}$ and for any $g \in \Gamma$, $\delta(g) = 1\otimes g$. Here $\delta : \mathcal{H} \to \Bbbk\Gamma\otimes\mathcal{H}$ is the coaction. Inspired by [1] we shall construct a Hopf algebra H such that the tensor categories of representations of H and $\mathcal{H}$ are equivalent.

Consider the bosonization $\mathcal{H}\#\Bbbk\Gamma$. The ideal I generated by elements $h\#1 - 1\#h$ for all $h \in \Gamma$ is a Hopf ideal. Define $H = \mathcal{H}\#\Bbbk\Gamma/I$. The class of an element $x\otimes g \in \mathcal{H}\#\Bbbk\Gamma$ in the quotient H will be denoted by $\overline{x\otimes g}$.

If $\mathcal{J} \in \mathcal{H}\otimes\mathcal{H}$ is a twist, define

$$J = \overline{(\mathrm{id}\otimes\delta)(\mathcal{J})\#1} = \overline{\mathcal{J}^1\#\mathcal{J}^2_{(-1)}\otimes\mathcal{J}^2_{(0)}\#1}. \tag{3.10}$$

We shall use the notation $\mathcal{J} = \mathcal{J}^1\otimes\mathcal{J}^2 = j^1\otimes j^2$.

Theorem 3.6 *The element $J \in H\otimes H$ is a twist. If $\mathcal{J}$ is Γ-invariant, that is $g\cdot\mathcal{J} = \mathcal{J}$ for all $g \in \Gamma$, then $\mathcal{H}^{\mathcal{J}}\#\Bbbk\Gamma/I \simeq H^J$.*

Proof Clearly J is invertible with inverse $J^{-1} = \overline{(\mathrm{id}\otimes\delta)(\mathcal{J}^{-1})\#1}$. Applying $(\mathrm{id}\otimes\delta\otimes\delta)$ to $(\Delta\otimes\mathrm{id})(\mathcal{J})(\mathcal{J}\otimes 1)$ we obtain

$$\begin{aligned}\mathcal{J}^1_{(1)}\big(\mathcal{J}^1_{(2)(-1)}\mathcal{J}^2_{(-2)}\big)\cdot j^1 \otimes \mathcal{J}^1_{(2)(-1)}\, j^2_{(-1)}\otimes\\ \otimes \mathcal{J}^1_{(2)(0)}\mathcal{J}^2_{(-1)}\cdot j^2_{(0)}\otimes\mathcal{J}^2_{(-1)}\otimes\mathcal{J}^2_{(0)}.\end{aligned}$$

Applying to this element $(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id})$, multiplying the fourth and second tensorands and using the cocommutativity of $\Bbbk\Gamma$ with obtain

$$\begin{aligned}\mathcal{J}^1_{(1)}\big(\mathcal{J}^1_{(2)(-2)}\mathcal{J}^2_{(-4)}\big)\cdot j^1 \otimes \mathcal{J}^1_{(2)(-1)}\mathcal{J}^2_{(-3)}\, j^2_{(-1)}\otimes\\ \otimes \mathcal{J}^1_{(2)(0)}\mathcal{J}^2_{(-2)}\cdot j^2_{(0)}\otimes\mathcal{J}^2_{(-1)}\otimes\mathcal{J}^2_{(0)}.\end{aligned} \tag{3.11}$$

On the other hand, applying $(\mathrm{id}\otimes\delta\otimes\delta)$ to $(\mathrm{id}\otimes\Delta)(\mathcal{J})(1\otimes\mathcal{J})$ we obtain

$$\mathcal{J}^1\otimes\mathcal{J}^2_{(1)(-1)}\, j^1_{(-1)}\otimes\mathcal{J}^2_{(1)(0)}\mathcal{J}^2_{(2)(-2)}\cdot j^1_{(0)}\otimes\mathcal{J}^2_{(2)(-1)}\, j^2_{(-1)}\otimes\mathcal{J}^2_{(2)(0)}\, j^2_{(0)}.$$

Again, applying to this element $(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id})$ and multiplying the fourth and second tensorands we obtain

$$\mathcal{J}^1\otimes\mathcal{J}^2_{(1)(-1)}\mathcal{J}^2_{(2)(-2)}\,j^1_{(-1)}\,j^2_{(-1)(1)}\otimes\mathcal{J}^2_{(1)(0)}\mathcal{J}^2_{(2)(-3)}\cdot j^1_{(0)}\otimes \\ \otimes\mathcal{J}^2_{(2)(-1)}\,j^2_{(-1)(2)}\otimes\mathcal{J}^2_{(2)(0)}\,j^2_{(0)}.$$

Using the cocommutativity of $\Bbbk\Gamma$ and that $\delta(\mathcal{J})=1\otimes\mathcal{J}$ the above element is equal to

$$\mathcal{J}^1\otimes\mathcal{J}^2_{(1)(-1)}\mathcal{J}^2_{(2)(-3)}\otimes\mathcal{J}^2_{(1)(0)}\mathcal{J}^2_{(2)(-2)}\cdot j^1_{(0)}\otimes\mathcal{J}^2_{(2)(-1)}\,j^2_{(-1)}\otimes\mathcal{J}^2_{(2)(0)}\,j^2_{(0)},$$

and since the coproduct is a morphism of $\Bbbk\Gamma$-comodules the above element is equal to

$$\mathcal{J}^1\otimes\mathcal{J}^2_{(1)}\otimes\mathcal{J}^2_{(0)(1)}\mathcal{J}^2_{(0)(2)(-2)}\cdot j^1_{(0)}\otimes\mathcal{J}^2_{(0)(2)(-1)}\,j^2_{(-1)}\otimes\mathcal{J}^2_{(2)(0)}\,j^2_{(0)}, \tag{3.12}$$

The element $(\Delta\otimes\mathrm{id})(J)(J\otimes 1)$ is equal to

$$(\Delta\otimes\mathrm{id})(\overline{\mathcal{J}^1\#\mathcal{J}^2_{(-1)}\otimes\mathcal{J}^2_{(0)}\#1})(J\otimes 1)= \\ =\overline{\mathcal{J}^1_{(1)}\big(\mathcal{J}^1_{(2)(-2)}\mathcal{J}^2_{(-4)}\big)\cdot j^1\#\mathcal{J}^1_{(2)(-1)}\mathcal{J}^2_{(-3)}\,j^2_{(-1)}}\otimes\overline{\mathcal{J}^2_{(2)(0)}\mathcal{J}^2_{(-2)}\cdot j^2_{(0)}\#\mathcal{J}^2_{(-1)}}\otimes\overline{\mathcal{J}^2_{(0)}\#1},$$

and $(\mathrm{id}\otimes\Delta)(J)(1\otimes J)$ equals

$$(\mathrm{id}\otimes\Delta)(\overline{\mathcal{J}^1\#\mathcal{J}^2_{(-1)}\otimes\mathcal{J}^2_{(0)}\#1})(1\otimes J)= \\ =\overline{\mathcal{J}^1\#\mathcal{J}^2_{(1)}}\otimes\overline{\mathcal{J}^2_{(0)(1)}\mathcal{J}^2_{(0)(2)(-2)}\cdot j^1_{(0)}\#\mathcal{J}^2_{(0)(2)(-1)}\,j^2_{(-1)}}\otimes\overline{\mathcal{J}^2_{(2)(0)}\,j^2_{(0)}\#1}.$$

Since $\mathcal{J}$ is a twist Eqs. 3.11 and 3.12 are equal, hence we conclude that $(\Delta\otimes\mathrm{id})(J)(J\otimes 1)=(\mathrm{id}\otimes\Delta)(J)(1\otimes J)$. It follows easily that the coproduct of $\mathcal{H}^{\mathcal{J}}\#\Bbbk\Gamma/I$ coincides with the coproduct of H^J. □

3.4 The Exponential Map in Quantum Linear Spaces

Let G be a finite group and $\theta\in\mathbb{N}$, $(g_1,\dots,g_\theta,\chi_1,\dots,\chi_\theta)$ be a datum for a quantum linear space and $V=V(g_1,\dots,g_\theta,\chi_1,\dots,\chi_\theta)$.

Let $\mathcal{D}=\{a_{ij}\in\Bbbk: 1\le i,j\le\theta, i\neq j\}\cup\{\xi_i\in\Bbbk: 1\le i\le\theta\}$ be a family of θ^2 scalars. We shall say that $\mathcal{D}$ is *compatible* with the quantum linear space V if

$$a_{ij}=0\ \text{ if }\ g_ig_j\neq 1,\qquad \xi_i=0\ \text{ if }\ g_i^{N_i}\neq 1. \tag{3.13}$$

Let $F\subseteq G$ be a subset. We shall say that the family of scalars $\mathcal{D}$ is *F-invariant* if

$$\chi_i(g)\chi_j(g)\,a_{ij}=a_{ij}, \tag{3.14}$$

$$\chi_i^{N_i}(g)\,\xi_i=\xi_i \quad \text{for all } g\in F. \tag{3.15}$$

In particular if $\mathcal{D}$ is compatible with V then it is Γ-invariant, which amounts to

$$a_{ij} = 0 \quad \text{if } \ q_{ik}q_{jk} \neq 1 \text{ for some } k = 1, \dots, \theta, \tag{3.16}$$

$$\xi_i = 0 \quad \text{if } \ q_{ij}^{N_i} \neq 1 \text{ for some } j = 1, \dots, \theta. \tag{3.17}$$

If $g \in G$ and $\mathcal{D}_1 = \{a_{ij} \in \Bbbk : 1 \leq i, j \leq \theta, i \neq j\} \cup \{\xi_i \in \Bbbk : 1 \leq i \leq \theta\}$, $\mathcal{D}_2 = \{a'_{ij} \in \Bbbk : 1 \leq i, j \leq \theta, i \neq j\} \cup \{\xi'_i \in \Bbbk : 1 \leq i \leq \theta\}$ are two families of scalars we will denote

$$\mathcal{D}_1 + \mathcal{D}_2 = \{a_{ij} + a'_{ij} \in \Bbbk\} \cup \{\xi_i + \xi'_i \in \Bbbk : 1 \leq i \leq \theta\},$$

$$g \cdot \mathcal{D}_1 = \{\chi_i(g)\chi_j(g)\, a_{ij} \in \Bbbk\} \cup \{\chi_i^{N_i}(g)\, \xi_i \in \Bbbk : 1 \leq i \leq \theta\}.$$

We shall say that a family of scalars $\mathcal{D}$ is *q-symmetric* if $a_{ij} = -q_{ij}a_{ji}$ for any $1 \leq i, j \leq \theta, i \neq j$. We shall denote

$$\widehat{\mathcal{D}} = \{b_{ij} \in \Bbbk : 1 \leq i < j \leq \theta, i \neq j\} \cup \{\xi_i \in \Bbbk : 1 \leq i \leq \theta\}$$

where $b_{ij} = q_{ij}a_{ji} - a_{ij}$. Clearly $\widehat{\mathcal{D}}$ is q-symmetric.

Define $B_{ij} = a_{ij}\, x_i \otimes x_j$, $J_{\xi_i} = 1\otimes 1 + \sum_{k=1}^{N_i-1} \frac{\xi_i}{(N_i-k)!_{q_i} k!_{q_i}}\, x_i^{N_i-k} \otimes x_i^k$ and

$$J_{\mathcal{D}} = \prod_{i=1}^{\theta} J_{\xi_i} \prod_{1 \leq i, j \leq \theta, i \neq j} \exp_{q_{ij}}(B_{ij}). \tag{3.18}$$

Theorem 3.7 *Let $\mathcal{D}$ be a compatible family of scalars with V. Then $J_{\mathcal{D}}$ is a twist for $\mathfrak{B}(V)$ in the category ${}_{\Gamma}^{\Gamma}\mathcal{YD}$.*

Proof It follows by Eq. 3.16 that

$$(1\otimes B_{ij})(\mathrm{id} \otimes \Delta(B_{kl})) = (\mathrm{id} \otimes \Delta(B_{kl}))(1\otimes B_{ij}),$$

$$(B_{ij}\otimes 1)(\Delta(B_{kl})\otimes \mathrm{id}) = (\Delta(B_{kl})\otimes \mathrm{id})(B_{ij}\otimes 1),$$

thus

$$(1\otimes \exp_{q_{ij}}(B_{ij}))(\mathrm{id} \otimes \Delta(\exp_{q_{kl}}(B_{kl}))) = (\mathrm{id} \otimes \exp_{q_{kl}}(B_{kl}))(1\otimes \exp_{q_{ij}}(B_{ij})),$$

$$(\exp_{q_{ij}}(B_{ij})\otimes 1)(\Delta(\exp_{q_{kl}}(B_{kl}))\otimes \mathrm{id}) = (\Delta(\exp_{q_{kl}}(B_{kl}))\otimes \mathrm{id})(\exp_{q_{ij}}(B_{ij})\otimes 1).$$

Using Lemma 3.2 we obtain that $\prod_{1 \leq i < j \leq \theta} \exp_{q_{ij}}(B_{ij})$ is a twist. It follows from Eq. 3.17 that

$$(1\otimes x_i^{N_i-k} \otimes x_i^k)(\mathrm{id} \otimes \Delta)(x_j^{N_j-a} \otimes x_j^a) = (\mathrm{id} \otimes \Delta)(x_j^{N_j-a} \otimes x_j^a)(1\otimes x_i^{N_i-k} \otimes x_i^k),$$

thus by Lemma 3.2 we conclude that $\prod_{i=1}^{\theta} J_{\xi_i}$ is a twist.

It follows from Eq. 3.17 that $(x_i^{N_i-k} \otimes x_i^k)(x_l \otimes x_j) = (x_l \otimes x_j)(x_i^{N_i-k} \otimes x_i^k)$, thus $\exp_{q_{lj}}(B_{lj})$ and J_{ξ_i} commute. Let $i, j, k = 1, \dots, \theta$, $1 \leq a \leq N_k$, then $(1\otimes x_k^{N_k-a} \otimes x_k^a)$ $(x_i \otimes 1 \otimes x_j + x_i \otimes x_j \otimes 1)$ equals

$$= x_i \otimes x_k^{N_k-a} \otimes x_k^a x_j + q_{kj}^a\, x_i \otimes x_k^{N_k-a} x_j \otimes x_k^a$$

$$= q_{kj}^a\, x_i \otimes x_k^{N_k-a} \otimes x_j x_k^a + q_{kj}^{N_k}\, x_i \otimes x_j x_k^{N_k-a} \otimes x_k^a.$$

On the other hand $(x_i\otimes 1\otimes x_j + x_i\otimes x_j\otimes 1)(1\otimes x_k^{N_k-a}\otimes x_k^a)$ equals

$$q_{jk}^{N_k-a}\ x_i\otimes x_k^{N_k-a}\otimes x_j x_k^a + x_i\otimes x_j x_k^{N_k-a}\otimes x_k^a.$$

Using Eq. 3.17, it follows that $(1\otimes x_k^{N_k-a}\otimes x_k^a)$ and $(\mathrm{id}\otimes\Delta)(x_i\otimes x_j)$ commute, hence $(1\otimes J_{\xi_i})$ and $(\mathrm{id}\otimes\Delta)(\exp_{q_{lj}}(B_{lj}))$ commute. Similarly we can prove that $(J_{\xi_i}\otimes 1)$ and $(\Delta\otimes\mathrm{id})(\exp_{q_{lj}}(B_{lj}))$ commute. Using again Lemma 3.2, it follows that $J_{\mathcal{D}}$ is a twist. □

Remark 3.8 For any $1\le k, l, s, t\le\theta$ we have that

$$\exp_{q_{kl}}(B_{kl})\exp_{q_{st}}(B_{st}) = \exp_{q_{st}}(B_{st})\exp_{q_{kl}}(B_{kl}),$$

$$\exp_{q_{kl}}(B_{kl})J_{\xi_j} = J_{\xi_j}\exp_{q_{kl}}(B_{kl}),\ \ J_{\xi_j}J_{\xi_t} = J_{\xi_t}J_{\xi_j}.$$

Remark 3.9 It would be interesting to study the exponential map for other types of Nichols algebras.

4 Hopf Algebras $A(V, G, W, F, J, \mathcal{D})$

Let G be a finite group and $\theta\in\mathbb{N}$, $(g_1,\dots,g_\theta,\chi_1,\dots,\chi_\theta)$ be a datum for a quantum linear space and $V = V(g_1,\dots,g_\theta,\chi_1,\dots,\chi_\theta)$. As before Γ is the Abelian group generated by $\{g_i : i = 1,\dots,\theta\}$. Note that Γ is contained in the center of G. Using ideas contained in [1] we shall construct Hopf algebras coming from twisting $\mathfrak{B}(V)\#\Bbbk G$.

By restriction V is an object in ${}_\Gamma^\Gamma\mathcal{Y}D$. The vector space $\mathfrak{B}(V)\otimes_\Bbbk\Bbbk G$ has a structure of braided Hopf algebra in ${}_\Gamma^\Gamma\mathcal{Y}D$ as follows. The coaction $\delta : \mathfrak{B}(V)\otimes_\Bbbk\Bbbk G\to\Bbbk\Gamma\otimes_\Bbbk\mathfrak{B}(V)\otimes_\Bbbk\Bbbk G$ and the action $\cdot : \Gamma\otimes_\Bbbk\mathfrak{B}(V)\otimes_\Bbbk\Bbbk G\to\mathfrak{B}(V)\otimes_\Bbbk\Bbbk G$ are determined by

$$\delta(v\otimes g) = v_{(-1)}\otimes v_{(0)}\otimes g,\quad h\cdot(v\otimes g) = h\cdot v\otimes g,$$

for all $v\in\mathfrak{B}(V)$, $g\in G$, $h\in\Gamma$. The product and coproduct in $\mathfrak{B}(V)\otimes_\Bbbk\Bbbk G$ are given by

$$(v\otimes g)(v'\otimes g') = vg\cdot v'\otimes gg',\quad \Delta(v\otimes g) = v_{(1)}\otimes g\otimes v_{(2)}\otimes g,$$

for all $v, v'\in\mathfrak{B}(V)$, $g, g'\in G$.

Let F be a subgroup of G such that $\Gamma\le F$, let $W\subseteq V$ be a subspace stable under the action of F and $W_g\subseteq V_g$ for all $g\in F$. In this case we can consider the braided Hopf algebra $\mathfrak{B}(W)\otimes\Bbbk F$. Let $\mathcal{D}$ be an F-invariant family of scalars for the quantum linear space W and let J be a twist of $\Bbbk F$.

Lemma 4.1 *The element $J_{\mathcal{D}}^1\otimes J^1\otimes J_{\mathcal{D}}^2\otimes J^2$ is a twist for the braided Hopf algebra $\mathfrak{B}(W)\otimes\Bbbk F$ in the category ${}_\Gamma^\Gamma\mathcal{Y}D$.*

Proof Both elements $J_{\mathcal{D}}^1\otimes 1\otimes J_{\mathcal{D}}^2\otimes 1$, $1\otimes J^1\otimes 1\otimes J^2$ are twists. Note that for any $g\in F$ we have that $g\cdot J_{\mathcal{D}} = J_{g\cdot\mathcal{D}}$, hence

$$(1\otimes g\otimes 1\otimes g)(J_{\mathcal{D}}^1\otimes 1\otimes J_{\mathcal{D}}^2\otimes 1) = J_{g\cdot\mathcal{D}}^1\otimes g\otimes J_{g\cdot\mathcal{D}}^2\otimes g.$$

Since $\mathcal{D}$ is F-invariant, it follows that Eqs. 3.5 and 3.6 are satisfied and by Lemma 3.2 $J_{\mathcal{D}}^1\otimes J^1\otimes J_{\mathcal{D}}^2\otimes J^2$ is a twist. □

Abusing the notation we shall denote by $J_{\mathcal{D}}J$ the twist $J^1_{\mathcal{D}}\otimes J^1\otimes J^2_{\mathcal{D}}\otimes J^2$.

Definition 4.2 Under the above assumptions we define the braided Hopf algebra

$$\mathcal{A}(V, G, W, F, J, \mathcal{D}) = (\mathfrak{B}(V)\otimes\Bbbk G)^{J_{\mathcal{D}}J}$$

in the category ${}^{\Gamma}_{\Gamma}\mathcal{Y}D$. Using the bosonization procedure we construct the Hopf algebra $\mathcal{A}(V, G, W, F, J, \mathcal{D})\#\Bbbk\Gamma$. The bilateral ideal I generated by elements $1\otimes h\#1 - 1\otimes 1\#h$ for all $h \in \Gamma$ is a Hopf ideal. Thus we define the Hopf algebra $A(V, G, W, F, J, \mathcal{D})$ as the quotient $\mathcal{A}(V, G, W, F, J, \mathcal{D})\#\Bbbk\Gamma/I$.

If $V = W$, $F = G$ and $J = 1\otimes 1$ we shall denote the Hopf algebra $A(V, G, V, \Gamma, J, \mathcal{D})$ simply by $A(V, G, \mathcal{D})$.

Remark 4.3 Note that if $\mathcal{D} = 0$, that is if $a_{ij} = 0 = \xi_i$ for all $1 \le i, j \le \theta$, then $J_{\mathcal{D}} = 1\otimes 1$ and $A(V, G, 0) = \mathfrak{B}(V)\#\Bbbk G$.

Corollary 4.4 *$A(V, G, W, F, J, \mathcal{D})$ is twist equivalent to $\mathfrak{B}(V)\#\Bbbk G$.*

Proof It follows from Theorem 3.6. □

Definition 4.5 [1] A tensor category is said to have the *Chevalley property* if the tensor product of simple objects is semisimple. A Hopf algebra H has the Chevalley property if the category of left H-modules does.

If H is a Hopf algebra with the Chevalley property and $J \in H\otimes H$ is a twist then H^J has the Chevalley property. Hence the families of Hopf algebras $A(V, G, W, F, J, \mathcal{D})$ have the Chevalley property.

Example 4.6 [1] Let G be a finite group and $u \in G$ be a central element of order 2. Let V be a G-module such that $u \cdot v = -v$ for all $v \in V$. The space V is a Yetter–Drinfeld module over G by declaring $V = V_u$, thus $\Gamma = \mathbb{Z}_2$. Let $\{x_1, \dots, x_\theta\}$ be a basis of V. In this case $q_{ij} = -1 = q_{ii}$ for all $1 \le i, j \le \theta$ and the Nichols algebra $\mathfrak{B}(V)$ is the exterior algebra $\wedge V$.

Let $\mathcal{D} = \{a_{ij} \in \Bbbk : 1 \le i, j \le \theta, i \ne j\} \cup \{\xi_i \in \Bbbk : 1 \le i \le \theta\}$ be a family of scalars. Note that $\mathcal{D}$ is automatically $\mathbb{Z}_2$-invariant. Define

$$B = \sum_{i\ne j} a_{ij}\, x_i\otimes x_j + \sum_{i=1}^{\theta} \xi_i\, x_i\otimes x_i \in V\otimes V.$$

Since $q_{ij} = -1$ then $J_{\mathcal{D}} = e^B$. Our definition of $A(V, G, \mathcal{D})$ coincides with the definition given in [1], see also [5], where this algebra is denoted by $A(V, G, B)$. Note, however, that in *loc. cit.* the authors assume that the element B is symmetric, that is $B \in S^2(V)$.

Let us develop a more particular example. Assume that V has a basis $\{x, y\}$. If $a \in \Bbbk$ denote $B_a = a\, x\otimes y - a\, y\otimes x$. In this case the twist e^{B_a} is gauge equivalent to the trivial twist $1\otimes 1$. Indeed if $c = e^{axy}$ then $e^{B_a} = \Delta(c)(c^{-1}\otimes c^{-1})$. Thus $A(V, G, B_a) \simeq \wedge V\#\Bbbk G$. The isomorphism is given by conjugation by c, thus one can not expect to apply [5, Prop. 2.1] in the general situation. Also $\mathcal{A}(V, G, B_a)$ is super cocommutative but B_a is not G-invariant hence [5, Corollary 5.3] is no longer true when B is not symmetric.

Example 4.7 Let G be a finite group with a character $\chi : G \to \Bbbk^\times$. Let $n \in \mathbb{N}$ and $\xi \in \Bbbk$. Let $u \in G$ be a central element of order n and $\chi(u) = q$ be a n-th primitive root of unity. Let V be the one-dimensional vector space generated by x with structure of Yetter–Drinfeld module over G given by

$$\delta(x) = u \otimes x, \quad g \cdot x = \chi(g)\, x, \ \text{ for all } g \in G.$$

The Nichols algebra of V is isomorphic to $\Bbbk[x]/(x^n)$. If $\mathcal{D}_\xi = \{\xi\}$ then $A(V, G, \mathcal{D}_\xi)$ is isomorphic to the algebra generated by elements $\{x, g : g \in G\}$ subject to relations

$$x^n = 0, \quad gx = \chi(g)\, xg, \ \text{ for all } g \in G.$$

The twist in this case is $J_\xi = 1 \otimes 1 + \sum_{k=1}^{n-1} \frac{\xi}{(n-k)!_q k!_q}\, x^{n-k} \otimes x^k$ and the coproduct is given by formulas

$$\Delta(x) = x \otimes 1 + u \otimes x, \ \ \Delta(g) = g \otimes g + \sum_{k=1}^{n-1} \frac{\xi(\chi^n(g) - 1)}{(n-k)!_q k!_q}\, x^{n-k} u^k g \otimes x^k g.$$

We shall prove later that if $\xi \neq 0$ and $\chi^n \neq 1$ then $A(V, G, \mathcal{D}_\xi)$ is not a pointed Hopf algebra and (unfortunately) $A(V, G, \mathcal{D}_\xi) \simeq A(V, G, \mathcal{D}_1)$.

4.1 Some Isomorphisms of $A(V, G, \mathcal{D})$

In this section we shall present some isomorphisms of Hopf algebras $A(V, G, \mathcal{D})$ in a similar way as in [5, Prop. 2.1].

Remark 4.8 The coproduct of the braided Hopf algebra $\mathfrak{B}(V) \otimes \Bbbk G$ is given by:

$$\Delta^{J_\mathcal{D}}(v \otimes g) = J_\mathcal{D}^{-1} \Delta(v \otimes 1)(1 \otimes g \otimes 1 \otimes g) J_\mathcal{D} = J_\mathcal{D}^{-1} J_{g \cdot \mathcal{D}} \Delta(v \otimes g),$$

for all $g \in G$, $v \in \mathfrak{B}(V)$. This equation follows since $J_\mathcal{D} \Delta(v \otimes 1) = \Delta(v \otimes 1) J_\mathcal{D}$ for all $v \in \mathfrak{B}(V)$. This implies that if $\mathcal{D}$ is G-invariant then $A(V, G, \mathcal{D}) \simeq \mathfrak{B}(V) \# \Bbbk G$. In particular if $G = \Gamma$ then $A(V, G, \mathcal{D}) \simeq \mathfrak{B}(V) \# \Bbbk G$.

Let G, G' be finite groups, V, V' be quantum linear spaces over Γ and Γ' respectively with basis $\{x_1, \ldots, x_\theta\}$ and $\{x'_1, \ldots, x'_\theta\}$ respectively. Let $\mathcal{D}, \mathcal{D}'$ be families of scalars such that $\mathcal{D}$ is Γ-invariant and $\mathcal{D}'$ is Γ'-invariant.

Proposition 4.9 *The Hopf algebras $A(V, G, \mathcal{D})$, $A(V', G', \mathcal{D}')$ are isomorphic provided there is a group isomorphism $\phi : G \to G'$ such that $\phi(\Gamma) = \Gamma'$ and a isomorhism $\eta : V \to \phi^*(V')$ of Yetter–Drinfeld modules over G such that $(\eta \otimes \eta)(J_\mathcal{D}) = J_{\mathcal{D}'} J_{\widetilde{\mathcal{D}}}$ where $\widetilde{\mathcal{D}}$ is G-invariant.*

Proof We shall prove that the braided Hopf algebras $\mathfrak{B}(V) \otimes \Bbbk G$, $\mathfrak{B}(V') \otimes \Bbbk G'$ are isomorphic. The map η can be extended to an algebra map $\eta : \mathfrak{B}(V) \to \mathfrak{B}(V')$. Define $\psi : \mathfrak{B}(V) \otimes \Bbbk G \to \mathfrak{B}(V') \otimes \Bbbk G'$ by

$$\psi(v \otimes g) = \eta(v) \otimes \phi(g), \ \text{ for all } v \in \mathfrak{B}(V), g \in G.$$

Clearly ψ is a bijective algebra morphism. If $v \in \mathfrak{B}(V)$, $g \in G$ then

$$\Delta^{J_{\mathcal{D}'}}(\psi(v \otimes g)) = J_{\mathcal{D}'}^{-1} J_{g\cdot\mathcal{D}'} \Delta(\psi(v \otimes g)).$$

On the other hand

$$\begin{aligned}(\psi\otimes\psi)\Delta^{J_{\mathcal{D}}}(v\otimes g) &= (\eta\otimes\eta)(J_{\mathcal{D}}^{-1} J_{g\cdot\mathcal{D}})(\psi\otimes\psi)\Delta(v\otimes g)\\ &= J_{\mathcal{D}'}^{-1} J_{\widetilde{\mathcal{D}}}^{-1} J_{\phi(g)\cdot\mathcal{D}'} J_{\phi(g)\cdot\widetilde{\mathcal{D}}}(\psi\otimes\psi)\Delta(v\otimes g)\\ &= J_{\mathcal{D}'}^{-1} J_{\phi(g)\cdot\mathcal{D}'} J_{\widetilde{\mathcal{D}}}^{-1} J_{\widetilde{\mathcal{D}}}(\psi\otimes\psi)\Delta(v\otimes g)\\ &= J_{\mathcal{D}'}^{-1} J_{g\cdot\mathcal{D}'}\Delta(\psi(v\otimes g)).\end{aligned}$$

The third equation follows from Remark 4.8. □

4.2 The Algebra Structure of $A(V, G, \mathcal{D})^*$

In this section we shall describe the algebra structure of $A(V, G, \mathcal{D})^*$ following very closely the proof given in [5]. As a consequence we shall give necessary and sufficient conditions for the Hopf algebra $A(V, G, \mathcal{D})$ to be pointed.

We shall keep the notation of the previous section. Let $S \subseteq G$ be a set of representative classes of G/Γ, that is $G = \bigcup_{s\in S} s\Gamma$. For any $s \in S$ define

$$A_s = \{\overline{v\otimes g\#1} : v \in \mathfrak{B}(V), g \in s\Gamma\}.$$

If $\mathcal{D} = \{d_{ij} \in \Bbbk : 1 \le i, j \le \theta, i \ne j\} \cup \{\xi_i \in \Bbbk : 1 \le i \le \theta\}$ is a family of scalars then we define $\mathfrak{R}(\mathcal{D}, \Gamma)$ as the algebra generated by elements $\mathbb{X}_1, \dots, \mathbb{X}_\theta, \gamma \in \Gamma$ subject to relations

$$\gamma\mathbb{X}_i = \chi_i(\gamma)\,\mathbb{X}_i\gamma, \quad \mathbb{X}_i^{N_i} = \xi_i 1, \quad \mathbb{X}_i\mathbb{X}_j - q_{ij}\,\mathbb{X}_j\mathbb{X}_i = d_{ij}1.$$

The following result seems to be well-known.

Lemma 4.10 *The algebra $\mathfrak{R}(\mathcal{D}, \Gamma)$ is basic if and only if $d_{ij} = 0 = \xi_i$ for all $0 \le i, j \le \theta$.*

The following result generalizes [5, Thm. 5.2].

Proposition 4.11 *The following hold:*

1. *For any $s \in S$ the space A_s is a subcoalgebra,*
2. *$A(V, G, \mathcal{D}) = \oplus_{s\in S} A_s$,*
3. *there is an isomorphism of algebras $A_s^* \simeq \mathfrak{R}(\widehat{\mathcal{D}} - s\cdot\widehat{\mathcal{D}}, \Gamma)$.*

Proof

1. It follows from the definition of the coproduct of $A(V, G, \mathcal{D})$ and the fact that the twist $J_{\mathcal{D}} \in \mathfrak{B}(V)\otimes\mathfrak{B}(V)$.
2. The product in A_s^* is described as follows. If $X, Y \in A_s^*$, $h \in A_s$ then

$$\langle X * Y, h\rangle = \langle X, J^{-1}hj^1\rangle\langle Y, J^{-2}hj^2\rangle, \tag{4.1}$$

where $J = j^1 \otimes j^2 = J^1_{\mathcal{D}} \otimes 1\#1 \otimes J^2_{\mathcal{D}} \otimes 1\#1$, $J^{-1} \otimes J^{-2} = J^{-1}$. A base of A_s is given by $\{\overline{x_1^{r_1} \dots x_\theta^{r_\theta} \otimes s\gamma\#1} : 1 \le r_i \le N_i, 1 \le i \le \theta, \gamma \in \Gamma\}$. A base for the dual space A_s^* is given by the family $\{X_1^{r_1} \dots X_\theta^{r_\theta} \gamma : 1 \le r_i \le N_i, 1 \le i \le \theta, g \in \Gamma\}$ where

$$\langle X_1^{r_1} \dots X_\theta^{r_\theta} g^*, \overline{x_1^{s_1} \dots x_\theta^{s_\theta} \otimes s\gamma\#1} \rangle = \begin{cases} \prod_{i=1}^{\theta} (r_i)!_{q_i} \langle g^*, \gamma \rangle & \text{if } r_i = s_i \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\langle g^*, g_i \rangle = \chi_i(g)$, for all $g_i \in \Gamma$. It is not difficult to verify that using the product (4.1) the following equations hold:

$$g^* * f^* = (gf)^*, \ \ g^* * X_i = \chi_i(g)\, X_i g^*, \quad X_i * X_j = X_i X_j + ((\chi_i \chi_j(s) - 1) a_{ij}) 1,$$

$$X_i * X_i^{N_i - 1} = (\xi_i - \chi_i^{N_i}(s) \xi_i) 1, \quad X_i^l * X_i^k = X_i^{k+l} \text{ for all } k + l < N_i - 1,$$

for all $g, f \in \Gamma$, $1 \le i, j \le \theta$. We shall give the proof of third equality, the proofs of the other equations are done in a completely similar way. Let $g \in \Gamma$ then

$$\begin{aligned} \langle X_i * X_j, \overline{1 \otimes sg\#1} \rangle &= \langle X_i, \overline{J^{-1}_{\mathcal{D}}\, j^1_{s \cdot \mathcal{D}} \otimes sg\#1} \rangle \langle X_j, \overline{J^{-2}_{\mathcal{D}}\, j^2_{s \cdot \mathcal{D}} \otimes sg\#1} \rangle \\ &= -a_{ij} \langle X_i, \overline{x_i \otimes sg\#1} \rangle \langle X_j, \overline{x_j \otimes sg\#1} \rangle + \\ &\quad + \chi_i \chi_j(s) a_{ij} \langle X_i, \overline{x_i \otimes sg\#1} \rangle \langle X_j, \overline{x_j \otimes sg\#1} \rangle \\ &= -a_{ij} + \chi_i \chi_j(s) a_{ij}. \end{aligned}$$

For the second equation we are using that the coefficient of the term $x_i \otimes x_j$ of $\exp_{q_{ij}}(B_{ij})^{-1}$ is $-a_{ij}$. Since $\langle X_i * X_j, \overline{x_i x_j \otimes 1\#1} \rangle = 1$ and $\langle X_i * X_j, \overline{v \otimes 1\#1} \rangle = 0$ for any $v \in \mathfrak{B}(V)$ different from 1 and $x_i x_j$ then the result follows.

Observe that since $X_i X_j = q_{ij}\, X_j X_i$ then

$$X_i * X_j - q_{ij}\, X_j * X_i = (q_{ij} a_{ji} - a_{ij} + \chi_i \chi_j(s) a_{ji} - q_{ij} \chi_i \chi_j(s) a_{ij}) 1.$$

Whence there is a well-defined projection $\mathfrak{R}(\widehat{\mathcal{D}} - s \cdot \widehat{\mathcal{D}}, \Gamma) \to A_s^*$ and since both algebras have the same dimension they must be isomorphic. □

We can generalize [5, Corollary 5.3].

Corollary 4.12 *The Hopf algebra $A(V, G, \mathcal{D})$ is pointed if and only if $\widehat{\mathcal{D}}$ is G-invariant. In particular if $\mathcal{D}$ is q-symmetric then $A(V, G, \mathcal{D})$ is pointed if and only if $\mathcal{D}$ is G-invariant if and only if $A(V, G, \mathcal{D}) \simeq \mathfrak{B}(V)\#\Bbbk G$.*

Proof $A(V, G, \mathcal{D})$ is pointed if and only if A_s is pointed for all $s \in S$ if and only if A_s^* is basic for all $s \in S$ if and only if $\mathfrak{R}(\widehat{\mathcal{D}} - s \cdot \widehat{\mathcal{D}}, \Gamma, s)$ is basic for all $s \in S$ if and only if $\widehat{\mathcal{D}}$ is s-invariant for all $s \in S$.

If $\mathcal{D}$ is q-symmetric then $a_{ij} - q_{ij} a_{ji} = 2a_{ij}$ thus $\widehat{\mathcal{D}}$ is G-invariant if and only if $\mathcal{D}$ is G-invariant. If $\mathcal{D}$ is G-invariant then by Remark 4.8 $A(V, G, \mathcal{D}) \simeq \mathfrak{B}(V)\#\Bbbk G$ thus $A(V, G, \mathcal{D})$ is pointed. □

Question 4.1 If $\widehat{\mathcal{D}}$ is G-invariant then $A(V, G, \mathcal{D}) \simeq \mathfrak{B}(V)\#\Bbbk G$?

As a last remark I would like to point out that the dual of the Hopf algebra $A(V, G, W, F, J, \mathcal{D})$ has coradical $\Bbbk^G$ and this family could be helpful to the study of Hopf algebras with coradical a Hopf subalgebra that has recently began in [4].

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