# MODULE CATEGORIES OVER EQUIVARIANTIZED TENSOR CATEGORIES

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Abstract. For a finite tensor category $\mathcal{C}$ and a Hopf monad $T : \mathcal{C} \to \mathcal{C}$ satisfying certain conditions we describe exact indecomposable left $\mathcal{C}^T$-module categories in terms of left $\mathcal{C}$-module categories and some extra data. We also give a 2-categorical interpretation of the process of equivariantization of module categories.

## Introduction

As is the case in the study of any algebraic structure, a fundamental rôle in the study of tensor categories is played by its "representations". The natural notion of representation of a tensor category $\mathcal{C}$ is that of a *module category* over $\mathcal{C}$. A (left) module category over a tensor category $\mathcal{C}$ is a $\Bbbk$-linear Abelian category $\mathcal{M}$ equipped with a $\mathcal{C}$-action, that is, an exact bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ endowed with functorial associativity and unit constraints which satisfy appropriate coherence conditions. This notion is recalled in Section 2, it can be regarded as a "categorification" of the notion of module over an algebra. Many papers have been devoted to the study of different aspects of module categories over a monoidal or tensor category in the last years.

In the context of finite tensor categories it is convenient to restrict the attention to the class of *exact* module categories: this class of module categories was introduced in [14], see also [11, Section 2.6]. By definition, a module category $\mathcal{M}$ is exact if it is finite and for any projective object $P$ of $\mathcal{C}$ and for any object $M$ of $\mathcal{M}$, the object $P\overline{\otimes}M$ is projective.

Examples of finite tensor categories over $\Bbbk$ are given by the categories of finite dimensional (co)modules over a finite dimensional Hopf algebra $H$ over $\Bbbk$. Module categories over such tensor categories have been investigated intensively for several different classes of Hopf algebras.

A natural generalization of a Hopf algebra is given by a Hopf monad, as introduced in [9], [6]. Let $\mathcal{C}$ be a tensor category over $\Bbbk$. A Hopf monad

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on $\mathcal{C}$ is a monad $T$ on $\mathcal{C}$ which is a comonoidal functor in a compatible way and such that certain associated fusion operators are invertible. If $T$ is a $\Bbbk$-linear right exact Hopf monad on a (finite) tensor category $\mathcal{C}$, then the Eilenberg-Moore category $\mathcal{C}^T$ of $T$-modules in $\mathcal{C}$ is also a (finite) tensor category over $\Bbbk$ and the forgetful functor $\mathcal{F}: \mathcal{C}^T \to \mathcal{C}$ is a tensor functor. This functor is in addition dominant if $T$ is a faithful endofunctor of $\mathcal{C}$.

The main goal of this paper is to give a description of exact indecomposable module categories over the tensor category $\mathcal{C}^T$ of $T$-modules in a (finite) tensor category $\mathcal{C}$, where $T$ is a $\Bbbk$-linear right exact faithful Hopf monad on $\mathcal{C}$.

In order to do this we introduce the notion of a *$T$-equivariant $\mathcal{C}$-module category*: this consists of the data $(\mathcal{M}, U, c)$, where $\mathcal{M}$ is a $\mathcal{C}$-module category, $U$ is a monad on $\mathcal{M}$, and $(U, c): \mathcal{M} \to \mathcal{M}(T)$ is a lax $\mathcal{C}$-module functor, such that the multiplication and unit morphisms of $U$ are morphisms of $\mathcal{C}$-module functors. See Definition 4.2. Here $\mathcal{M}(T)$ is a natural lax $\mathcal{C}$-module category arising from $\mathcal{M}$ and the lax comonoidal functor $T$.

We show in Theorem 4.9 that if $\mathcal{M}$ is a $T$-equivariant $\mathcal{C}$-module category then the category $\mathcal{M}^U$ is a $\mathcal{C}^T$-module category. We also establish some functorial properties of this assignment and, in particular, give conditions in order that $\mathcal{M}^U$ be a simple module category in terms of $\mathcal{M}$ and $U$.

Our main result states that if $T$ is a right exact faithful Hopf monad on $\mathcal{C}$ and $\mathcal{M}$ is an exact indecomposable $\mathcal{C}^T$-module category, then there exists a $T$-equivariant indecomposable exact $\mathcal{C}$-module category $\mathcal{N}$ with simple and exact equivariant structure $U: \mathcal{N} \to \mathcal{N}$ such that $\mathcal{M} \simeq \mathcal{N}^U$ as $\mathcal{C}^T$-module categories. See Theorem 4.12 and Corollary 4.13. This result can be thought of as an extension of some of the results obtained in the context of module categories over representations of finite dimensional Hopf algebras in [2] (see Example 4.6).

One of the tools in the proof of the main result is an investigation of the relation between module categories over the category $\mathcal{C} \rtimes T = \operatorname{End}_{\mathcal{C}^T}(\mathcal{C})$ of $\mathcal{C}^T$-module endofunctors of $\mathcal{C}$ and $T$-equivariant $\mathcal{C}$-module categories. We show in Theorem 4.12 that every $\mathcal{C} \rtimes T$-module category $\mathcal{N}$ has a natural structure of a $T$-equivariant module category; in fact, the Hopf monad $T$ can be regarded as an algebra in $\mathcal{C} \rtimes T$, and the relevant data $U: \mathcal{N} \to \mathcal{N}(T)$ for the $T$-equivariance of $\mathcal{N}$ is provided by the action of $T$ on $\mathcal{N}$.

Recall that, for a given tensor category $\mathcal{C}$, $\mathcal{C}$-module categories, (lax) $\mathcal{C}$-module functors and $\mathcal{C}$-module natural transformations constitute a 2-category, that we denote ${}_{\mathcal{C}}\mathrm{Mod}$ (respectively, ${}_{\mathcal{C}}\mathrm{Mod}^{lax}$). We show that the assignment $\mathcal{M} \mapsto \mathcal{M}(T)$ extends to a 2-monad $\mathfrak{T}$ on ${}_{\mathcal{C}}\mathrm{Mod}^{lax}$, and there is a 2-equivalence of 2-categories

$$ {}_{\mathcal{C}}\mathrm{EqMod} \simeq ({}_{\mathcal{C}}\mathrm{Mod}^{lax})^{\mathfrak{T}}, $$

where ${}_{\mathcal{C}}\mathrm{EqMod}$ is the 2-category of $T$-equivariant lax $\mathcal{C}$-module categories. This is proved in Proposition 6.8, and gives a 2-categorical interpretation of the process of equivariantization of module categories.

As an application we give a description of module categories over Hopf algebroids, as defined for instance in [4], [5], [15]. We show in Theorem 5.8 that under the assumption that the basis of the Hopf algebroid $H$ is simple (which guarantees that $H$- mod is indeed a tensor category), then every exact indecomposable module category over $H$- mod is equivalent to $K$- mod for some $H$-simple left $H$-comodule algebra $K$.

We then consider the special situation where the Hopf monad $T$ is *normal*, according to the definition given in [7]: recall that this means that $T$ restricts to a Hopf monad on the trivial subcategory of $\mathcal{C}$. Such Hopf monad gives rise to an exact sequence of tensor categories comod-$H \longrightarrow \mathcal{C}^T \longrightarrow \mathcal{C}$, where $H$ is the induced Hopf algebra of $T$, which is finite dimensional. In this context we study the category $\mathcal{C} \rtimes T$ and show that it is (reversed) equivalent as a $\Bbbk$-linear category to the Deligne tensor product $H$- mod $\boxtimes\, \mathcal{C}$.

The paper is organized as follows. In Sections 1 and 2 we recall the definitions and main basic features of tensor categories and their module categories and Hopf monads on tensor categories and the associated categories of modules, respectively. In particular, given a Hopf monad $T$ on a tensor category $\mathcal{C}$, we discuss in this section the Morita dual of the category $\mathcal{C}^T$ with respect to its canonical module category $\mathcal{C}$. In Section 4 we study module categories over the category $\mathcal{C}^T$. Theorem 4.12 and Corollary 4.13 are proved in this section; the notions of $T$-equivariant module category and simple $T$-equivariant module category are also introduced here. Section 5 presents an application of the results in the previous section to the category of representations of a finite-dimensional Hopf algebroid. In Section 6 we give a 2-categorical interpretation of equivariantization of module categories. Finally in Section 7 we discuss the case where the Hopf monad $T$ is normal and give some examples.

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## 1. Preliminaries and Notation

We shall work over an algebraically closed field $\Bbbk$ of characteristic 0. All vector spaces and algebras will be over $\Bbbk$. If $A$ is an algebra then $A$- mod and mod-$A$ will denote the category of finite-dimensional left and right $A$-modules, respectively. The category of finite d imensional $A$-bimodules will be indicated by $A$- mod-$A$. Similarly, if $C$ is a coalgebra, then comod-$C$ will denote the category of finite-dimensional left $C$-comodules.

We refer the reader to [19] for the basic notions regarding categories and functors assumed throughout. Recall a category $\mathcal{C}$ is called *locally small* if for any objects $X, Y$ of $\mathcal{C}$, the class of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set; $\mathcal{C}$ is called *small* if it is locally small and its class of objects is a set. The category $\mathcal{C}$ is called *essentially small* (respetively, *essentially locally small*) if it is equivalent to a small (respetively, locally small) category. Unless otherwise stated all categories considered in this paper will be assumed to be essentially small.

1.1. **Tensor categories.** A *tensor category over* $\Bbbk$ is a $\Bbbk$-linear Abelian rigid monoidal category $\mathcal{C}$ such that the tensor product functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is $\Bbbk$-linear in each variable, and the following conditions hold:

- Hom spaces are finite dimensional,
- all objects of $\mathcal{C}$ have finite length,
- the unit object $\mathbf{1}$ is simple.

A *finite tensor category* [14] is a tensor category that has a finite number of isomorphism classes of simple objects and every simple object has a projective cover. Hereafter all tensor categories will be considered over $\Bbbk$ and all functors between them will be assumed to be $\Bbbk$-linear.

Observe that if $\mathcal{C}$ is a tensor category over $\Bbbk$, then it follows by rigidity that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bi-exact.

If $\mathcal{C}$ is a tensor category, we shall denote by $\mathcal{C}^{\mathrm{rev}}$ the tensor category whose underlying Abelian category is $\mathcal{C}$, endowed with the opposite tensor product:

$$X \otimes^{\mathrm{rev}} Y = Y \otimes X, \quad X, Y \in \mathcal{C}.$$

and associativity constraint $a^{\mathrm{rev}}_{X,Y,Z} = a^{-1}_{Z,Y,X}$, $X, Y, Z \in \mathcal{C}$, where $a$ is the associativity constraint of $\mathcal{C}$. Throughout this paper all tensor categories will be assumed to be strict, unless explicitly mentioned.

1.2. **Tensor functors.** A *tensor functor* from a tensor category $\mathcal{C}$ to a tensor category $\mathcal{D}$ is a $\Bbbk$-linear exact strong monoidal functor $F : \mathcal{C} \to \mathcal{D}$. A tensor functor preserves duals and is automatically faithful.

A tensor functor $F : \mathcal{C} \to \mathcal{D}$ is called *dominant* if it satisfies any of the following equivalent conditions ([7, Lemma 3.1]):

(i) Any object $Y$ of $\mathcal{D}$ is a subobject of $F(X)$ for some object $X$ of $\mathcal{C}$;
(ii) Any object $Y$ of $\mathcal{D}$ is a quotient of $F(X)$ for some object $X$ of $\mathcal{C}$;
(iii) The Pro-adjoint of $F$ is faithful;
(iv) The Ind-adjoint of $F$ is faithful.

On the other hand, $F$ is called *surjective* if any object of $\mathcal{D}$ is a subquotient of $F(X)$ for some $X$ of $\mathcal{C}$ [14, Definition 2.4]. In particular, every dominant tensor functor is surjective.

Let $F : \mathcal{C} \to \mathcal{D}$ be a dominant tensor functor between tensor categories $\mathcal{C}, \mathcal{D}$. Suppose that $F$ admits a right adjoint $R : \mathcal{D} \to \mathcal{C}$. Note that $R$ is automatically faithful since $F$ is dominant. By [7, Proposition 6.1], $A =$

$R(\mathbf{1})$ is equipped with the structure of a central commutative algebra $(A, \sigma)$ in $\mathcal{Z}(\mathcal{C})$.

Assume in addition that the right adjoint $R : \mathcal{D} \to \mathcal{C}$ of $F$ is exact. In this case, $F$ is called a *perfect* tensor functor. Then the category $\mathcal{C}_A$ of right $A$-modules in $\mathcal{C}$ is a tensor category with the monoidal structure induced by $\otimes_A$ and the half-braiding $\sigma$, and the functor $F$ is equivalent over $\mathcal{C}_A$ to the free module functor $F_A : \mathcal{C} \to \mathcal{C}_A$, $X \mapsto X \otimes A$. That is, there is an equivalence of tensor categories $K : \mathcal{D} \to \mathcal{C}_A$ such that $KF = F_A$.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are finite tensor categories and let $F : \mathcal{C} \to \mathcal{D}$ be a dominant tensor functor. Then $F$ admits (left and right) adjoints. Furthermore, if $\mathcal{D}$ is a fusion category then the right adjoint of $F$ is exact and therefore $F$ is a perfect tensor functor. See [8, Subsection 2.2].

## 2. Module categories

A (left) *module category* over a tensor category $\mathcal{C}$ is a locally finite $\Bbbk$-linear Abelian category $\mathcal{M}$ equipped with a bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, that we will sometimes refer as the *action*, which is $\Bbbk$-bilinear and bi-exact, endowed with natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y)\overline{\otimes} M \to X\overline{\otimes}(Y\overline{\otimes} M)$, $\ell_M : \mathbf{1}\overline{\otimes} M \to M$. These isomorphisms are subject to the following conditions:

$$m_{X,Y,Z\otimes M}\, m_{X\otimes Y,Z,M} = (\mathrm{id}_X \otimes m_{Y,Z,M})\, m_{X,Y\otimes Z,M}, \tag{2.1}$$

$$(\mathrm{id}_X \otimes l_M) m_{X,\mathbf{1},M} = \mathrm{id}_{X\overline{\otimes} M}. \tag{2.2}$$

See [11, Subsection 2.3]. Sometimes we shall also say that $\mathcal{M}$ is a $\mathcal{C}$-*module*.

We shall say that $\mathcal{M}$ is a *lax* $\mathcal{C}$-module when possibly the associativity and unit maps $m_{X,Y,M}$ and $\ell_M$ are not necessarily isomorphisms.

A *module functor* between module categories $\mathcal{M}$ and $\mathcal{M}'$ over a tensor category $\mathcal{C}$ is a pair $(F, c)$, where

- $F : \mathcal{M} \to \mathcal{M}'$ is a left exact functor;
- $c$ is a natural isomorphism: $c_{X,M} : F(X\overline{\otimes} M) \to X\overline{\otimes} F(M)$, $X \in \mathcal{C}$, $M \in \mathcal{M}$, such that for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$:

$$(\mathrm{id}_X \overline{\otimes} c_{Y,M}) c_{X,Y\overline{\otimes} M} F(m_{X,Y,M}) = m_{X,Y,F(M)}\, c_{X\otimes Y,M} \tag{2.3}$$

$$\ell_{F(M)}\, c_{\mathbf{1},M} = F(\ell_M). \tag{2.4}$$

If the maps $c_{X,M}$ satisfying (2.3) and (2.4) are not necessarily isomorphisms, the pair $(F, c)$ will be called a *lax module functor*.

There is a composition of module functors: if $\mathcal{M}''$ is another module category and $(G, d) : \mathcal{M}' \to \mathcal{M}''$ is another module functor then the composition

$$(G \circ F, e) : \mathcal{M} \to \mathcal{M}'', \qquad e_{X,M} = d_{X,F(M)} \circ G(c_{X,M}), \tag{2.5}$$

is also a module functor.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be $\mathcal{C}$-modules. We denote by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ the category whose objects are module functors $(F, c)$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. A morphism between $(F, c)$ and $(G, d) \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is a natural transformation $\alpha : F \to G$ such that for any $X \in \mathcal{C}$, $M \in \mathcal{M}_1$:

$$d_{X,M}\alpha_{X\overline{\otimes}M} = (\mathrm{id}_X\overline{\otimes}\alpha_M)c_{X,M}. \tag{2.6}$$

We shall also say that $\alpha : F \to G$ is a *$\mathcal{C}$-module transformation.*

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exist module functors $F : \mathcal{M}_1 \to \mathcal{M}_2$ and $G : \mathcal{M}_2 \to \mathcal{M}_1$ and natural isomorphisms $\mathrm{id}_{\mathcal{M}_1} \to F \circ G$, $\mathrm{id}_{\mathcal{M}_2} \to G \circ F$ that satisfy (2.6).

The direct sum of two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ is the $\Bbbk$-linear category $\mathcal{M}_1 \times \mathcal{M}_2$ with coordinate-wise module structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories.

Let $\mathcal{C}$ be a finite tensor category. Recall from [14] that a module category $\mathcal{M}$ is *exact* if $\mathcal{M}$ is finite and for any projective object $P \in \mathcal{C}$ the object $P\overline{\otimes}M$ is projective in $\mathcal{M}$, for all $M \in \mathcal{M}$.

**Example 2.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a dominant tensor functor between finite tensor categories $\mathcal{C}, \mathcal{D}$. Then the functor $\overline{\otimes} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$, given by $X\overline{\otimes}Y = F(X)\otimes Y$, for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$, endows $\mathcal{D}$ with a structure of an indecomposable $\mathcal{C}$-module category. Since $F$ is dominant (thus surjective), then $\mathcal{D}$ is in fact an exact module category over $\mathcal{C}$; see [14, Example 3.3 (i)].

A submodule category of a $\mathcal{C}$-module $\mathcal{M}$ is a Serre subcategory $\mathcal{N}$ (that is, $\mathcal{N}$ is a full subcategory such that for every short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{M}$, the object $X$ belongs to $\mathcal{N}$ if and only if $X'$ and $X''$ belong to $\mathcal{N}$), such that the inclusion functor $\mathcal{N} \to \mathcal{M}$ is a module functor. A module category is *simple* if it has no non-trivial submodule categories. It is known that for exact module categories the notions of indecomposability and simplicity are equivalent.

*Remark* 2.2. If $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$ is an indecomposable exact $\mathcal{C}$-module, the dual category $\mathcal{C}^*_{\mathcal{M}} = \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is again a finite tensor category [14]. It is shown in [14, Theorem 3.31] that there is a bijective correspondence between equivalence classes of exact indecomposable left module categories over $\mathcal{C}$ and over $\mathcal{C}^*_{\mathcal{M}}$. The correspondence assigns to a left $\mathcal{C}$-module category $\mathcal{N}$ the left $\mathcal{C}^*_{\mathcal{M}}$-module category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$. This fact implies that there is a bijective correspondence between equivalence classes of exact indecomposable left $\mathcal{C}$-module categories and equivalence classes of exact indecomposable *right* $\mathcal{C}^*_{\mathcal{M}}$-module categories, which assigns to every left $\mathcal{C}$-module category $\mathcal{N}$ the right $\mathcal{C}^*_{\mathcal{M}}$-module category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$.

Let $(F, \xi, \phi) : \mathcal{C} \to \tilde{\mathcal{C}}$ be a comonoidal functor and let $(\mathcal{M}, \overline{\otimes}, m)$ be a module category over $\tilde{\mathcal{C}}$. We shall denote by $\mathcal{M}(F)$ the lax module category over $\mathcal{C}$ with underlying Abelian category $\mathcal{M}$ and action, associativity and unit morphisms defined, respectively, by

$$X \overline{\otimes}^F M = F(X) \overline{\otimes} M,$$

$$m^F_{X,Y,M} = m_{F(X),F(Y),M}(\xi_{X,Y} \overline{\otimes} \operatorname{id}_M), \quad l^F_M = l_M(\phi \overline{\otimes} \operatorname{id}_M),$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$.

## 3. Hopf monads and tensor categories

3.1. **Hopf Monads.** Let $\mathcal{C}$ be a category. A *monad* on $\mathcal{C}$ is an algebra in the strict monoidal category $\operatorname{End}(\mathcal{C})$, that is, a triple $(T, \mu, \eta)$ where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\mu : T^2 \to T$ and $\eta : \operatorname{Id} \to T$ are natural transformations such that

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)}, \qquad \mu_X \eta_{T(X)} = \operatorname{id}_{T(X)} = \mu_X T(\eta_X). \tag{3.1}$$

Let $(T, \mu, \eta)$ be a monad on a category $\mathcal{C}$. An *action* of $T$ on an object $X$ of $\mathcal{C}$ is a morphism $r : T(X) \to X$ in $\mathcal{C}$ such that:

$$rT(r) = r\mu_X \quad \text{and} \quad r\eta_X = \operatorname{id}_X. \tag{3.2}$$

The pair $(X, r)$ is called a *$T$-module*. Given two $T$-modules $(X, r)$ and $(Y, s)$ in $\mathcal{C}$, a morphism of $T$-modules from $(X, r)$ to $(Y, s)$ is a morphism $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ such that $f \circ r = s \circ T(f)$. The category of $T$-modules will be denoted by $\mathcal{C}^T$. We shall denote by $\mathcal{F} = \mathcal{F}_T : \mathcal{C}^T \to \mathcal{C}$ the forgetful functor defined by $\mathcal{F}(X, r) = X$.

Let $(T_1, \mu_1, \eta_1)$, $(T_2, \mu_2, \eta_2)$ be monads on $\mathcal{C}$. A *morphism of monads* $\alpha : (T_1, \mu_1, \eta_1) \to (T_2, \mu_2, \eta_2)$ (below indicated by $\alpha : T_1 \to T_2$) is a natural transformation $\alpha : T_1 \to T_2$ such that

$$\alpha_X \mu_{1X} = \mu_{2X} \alpha_{T_2(X)} T(\alpha_X), \qquad \alpha_X \eta_{1X} = \eta_{2X}, \qquad \text{for all } X \in \mathcal{C}.$$

In view of [9, Lemma 1.7], a morphism of monads $\alpha : T_1 \to T_2$ induces a functor $\alpha^* : \mathcal{C}^{T_2} \to \mathcal{C}^{T_1}$ over $\mathcal{C}$, in the form $\alpha^*(X, r) = (X, r \circ \alpha_X)$, such that $\mathcal{F}_{T_1} \alpha^* = \mathcal{F}_{T_2}$. Furthermore, every such functor is of the form $\alpha^*$ for some morphism of monads $\alpha : T_1 \to T_2$.

A *bimonad* on a monoidal category $\mathcal{C}$ is a monad $(T, \mu, \eta)$ on $\mathcal{C}$ such that the functor $T$ is equipped with a comonoidal structure and the natural transformations $\mu$ and $\eta$ are comonoidal transformations. This means that there is a natural transformation $\xi_{X,Y} : T(X \otimes Y) \to T(X) \otimes T(Y)$ and a morphism $\phi : T(\mathbf{1}) \to \mathbf{1}$ in $\mathcal{C}$ such that the following conditions hold:

$$(\operatorname{id}_{T(X)} \otimes \xi_{Y,Z}) \xi_{X, Y \otimes Z} = (\xi_{X,Y} \otimes \operatorname{id}_{T(Z)}) \xi_{X \otimes Y, Z}, \tag{3.3}$$

$$(\operatorname{id}_{T(X)} \otimes \phi) \xi_{X, \mathbf{1}} = \operatorname{id}_{T(X)} = (\phi \otimes \operatorname{id}_{T(X)}) \xi_{\mathbf{1}, X}, \tag{3.4}$$

$$\xi_{X,Y} \mu_{X \otimes Y} = (\mu_X \otimes \mu_Y) \xi_{T(X), T(Y)} T(\xi_{X,Y}), \tag{3.5}$$

$$\phi \mu_{\mathbf{1}} = \phi T(\phi), \quad \xi_{X,Y} \eta_{X \otimes Y} = \eta_X \otimes \eta_Y, \quad \phi \eta_{\mathbf{1}} = \operatorname{id}_{\mathbf{1}}. \tag{3.6}$$

*Remark* 3.1. It is not required that $T$ is a strong comonoidal functor, meaning that $\xi_{X,Y}$ might not be isomorphisms.

If $T$ is a bimonad on the monoidal category $\mathcal{C}$, then $\mathcal{C}^T$ is a monoidal category with tensor product

$$(X,r)\otimes(Y,s) = (X\otimes Y, (r\otimes s)\xi_{X,Y}),$$

for all $(X,r),(Y,s)\in\mathcal{C}^T$. The unit object of $\mathcal{C}^T$ is $(\mathbf{1},\phi)$. For more details see [17], [9].

Note that in this case the forgetful functor $\mathcal{F}:\mathcal{C}^T\to\mathcal{C}$ is a strict strong monoidal functor. The functor $\mathcal{F}$ has a left adjoint $\mathcal{L}:\mathcal{C}\to\mathcal{C}^T$, such that $\mathcal{L}(X)=(T(X),\mu_X)$, for every object $X$ of $\mathcal{C}$. The unit and counit of the adjunction $(\mathcal{L},\mathcal{F})$ are given, respectively, by $\eta_X : X\to T(X)$ and $\epsilon_{(M,r)} = r : (T(M),\mu_M)\to(M,r)$, for all $X\in\mathcal{C}$, $(M,r)\in\mathcal{C}^T$.

The left adjoint $\mathcal{L}$ is a comonoidal functor with comonoidal structure given by

$$\mathcal{L}_2(X,Y) = \xi_{X,Y} : (T(X\otimes Y),\mu_{X\otimes Y})\to (T(X),\mu_X)\otimes(T(Y),\mu_Y), \tag{3.7}$$

for all $X,Y\in\mathcal{C}$, and $\mathcal{L}_0=\phi : (T(\mathbf{1}),\mu_{\mathbf{1}})\to(\mathbf{1},\phi)$. Moreover, the pair $(\mathcal{L}:\mathcal{C}\to\mathcal{C}^T, \mathcal{F}:\mathcal{C}^T\to\mathcal{C})$ is a *comonoidal adjunction* in the sense of [6, Subsection 2.5]; see [6, Example 2.4].

If $(T,\xi,\phi)$ is bimonad on the monoidal category $\mathcal{C}$, then $(\overline{T},\overline{\xi},\overline{\phi})$ is a bimonad on $\mathcal{C}^{\rm rev}$, where $\overline{T}=T$ as monads, $\overline{\phi}=\phi$, and $\overline{\xi}_{X,Y}=\xi_{Y,X}$ for all $X,Y\in\mathcal{C}$.

**Lemma 3.2.** *The identity functor induces a strict equivalence of monoidal categories* $(\mathcal{C}^{\rm rev})^{\overline{T}}\simeq(\mathcal{C}^T)^{\rm rev}$.

*Proof.* The proof is straightforward. □

Let $\mathcal{C}$ be a monoidal category. A bimonad $T$ on $\mathcal{C}$ is a *Hopf monad* if the *fusion operators* $H^l$ and $H^r$ defined, for all $X,Y\in\mathcal{C}$, by

$$H^l_{X,Y} := (\mathrm{id}_{T(X)}\otimes\mu_Y)\,\xi_{X,T(Y)} : T(X\otimes T(Y))\to T(X)\otimes T(Y), \tag{3.8}$$

$$H^r_{X,Y} := (\mu_X\otimes\mathrm{id}_{T(Y)})\,\xi_{T(X),Y} : T(T(X)\otimes Y)\to T(X)\otimes T(Y), \tag{3.9}$$

are isomorphisms [6, Subsection 2.7]. If $\mathcal{C}$ is a rigid monoidal category and $T$ is a Hopf monad on $\mathcal{C}$ then the category $\mathcal{C}^T$ is rigid [6, Subsection 3.4].

*Remark* 3.3. Suppose that $T$ is a Hopf monad on $\mathcal{C}$. It follows from [6, Theorem 2.15] that $(\mathcal{L},\mathcal{F})$ is indeed a *Hopf adjunction*, that is, the left and right Hopf operators $\mathbb{H}^l$ and $\mathbb{H}^r$ defined, for every $Y\in\mathcal{C}$, $(M,r)\in\mathcal{C}^T$, by

$$\mathbb{H}^l = (\mathrm{id}_{\mathcal{L}(Y)}\otimes r)\,\xi_{Y,M} : \mathcal{L}(Y\otimes M)\to\mathcal{L}(Y)\otimes\mathcal{L}(M), \tag{3.10}$$

$$\mathbb{H}^r = (r\otimes\mathrm{id}_{\mathcal{L}(Y)})\,\xi_{M,Y} : \mathcal{L}(M\otimes Y)\to\mathcal{L}(M)\otimes\mathcal{L}(Y), \tag{3.11}$$

are isomorphisms.

*Remark* 3.4. Suppose that $\mathcal{C}$ is a $\Bbbk$-linear Abelian category and $T$ is a $\Bbbk$-linear right exact monad on $\mathcal{C}$. Then the category $\mathcal{C}^T$ is $\Bbbk$-linear Abelian and the forgetful functor $\mathcal{F}:\mathcal{C}^T \to \mathcal{C}$ is $\Bbbk$-linear exact. In this case, if $\mathcal{C}$ is a tensor category over $\Bbbk$, then $\mathcal{C}^T$ is a tensor category over $\Bbbk$ and the forgetful functor $\mathcal{F}:\mathcal{C}^T \to \mathcal{C}$ is a tensor functor [7, Proposition 2.3].

**Lemma 3.5.** *Suppose that $\mathcal{C}$ is a finite tensor category. Then so is $\mathcal{C}^T$.*

*Proof.* The assumption that the tensor category $\mathcal{C}$ is finite is equivalent to the assumption that it has a projective generator $P$, that is, an object $P$ of $\mathcal{C}$ such that the functor $\operatorname{Hom}_{\mathcal{C}}(P,-)$ is faithful exact. Let $\mathcal{L}:\mathcal{C}\to\mathcal{C}^T$ be the left adjoint of the forgetful functor $\mathcal{F}:\mathcal{C}^T\to\mathcal{C}$. By adjointness, we obtain a natural isomorphism $\operatorname{Hom}_{\mathcal{C}^T}(\mathcal{L}(P),-)\simeq \operatorname{Hom}_{\mathcal{C}}(P,-)\circ\mathcal{F}$. Since $\mathcal{F}$ is faithful and exact, then $\operatorname{Hom}_{\mathcal{C}^T}(\mathcal{L}(P),-)$ is faithful exact, that is, $\mathcal{L}(P)$ is a projective generator of $\mathcal{C}^T$. Thus $\mathcal{C}^T$ is a finite tensor category as claimed. □

**Example 3.6.** Let $G$ be a finite group and let $\rho:\underline{G}\to\underline{\operatorname{Aut}}_{\otimes}(\mathcal{C})$ be an action of $G$ on $\mathcal{C}$ by tensor autoequivalences. In other words, for any $g\in G$ we have a tensor functor $(\rho^g,\zeta_g):\mathcal{C}\to\mathcal{C}$, and for any $g,h\in G$, there are natural isomorphisms of tensor functors $\gamma_{g,h}:\rho^g\circ\rho^h\to\rho^{gh}$ and $\rho_0:\operatorname{id}_{\mathcal{C}}\to\rho^e$. Associated to such an action there is a tensor category $\mathcal{C}^G$, called the *equivariantization* of $\mathcal{C}$ under the action $\rho$, endowed with a canonical dominant tensor functor $\mathcal{C}^G\to\mathcal{C}$.

It was shown in [7, Theorem 5.21] that the action $\rho$ induces a Hopf monad $T^\rho$ on $\mathcal{C}$ in such a way that $\mathcal{C}^{T^\rho}\simeq\mathcal{C}^G$ as tensor categories over $\mathcal{C}$. The Hopf monad $T^\rho$ is defined in the form $T^\rho(X)=\bigoplus_{g\in G}\rho^g(X)$, with multiplication $\mu:(T^\rho)^2=\bigoplus_{g,h}\rho^g\rho^h\to T^\rho=\bigoplus_{t\in G}\rho^t$ and unit $\eta:\operatorname{id}_{\mathcal{C}}\to T^\rho=\bigoplus_{g\in G}\rho^g$, defined componentwise by the morphisms $\gamma_{g,h}:\rho^g\rho^h\to\rho^{gh}$, and by $\rho_0:\operatorname{id}_{\mathcal{C}}\to\rho^e$, respectively. The comonoidal structure morphisms $\xi_{X,Y}:\bigoplus_{g\in G}\rho^g(X\otimes Y)\to\bigoplus_{s,t\in G}\rho^s(X)\otimes\rho^t(Y)$, and $\phi:\bigoplus_{g\in G}\rho^g(\mathbf{1})\to\mathbf{1}$, are defined componentwise by the strong comonoidal structure of the tensor functors $\rho^g$.

The Hopf monad $T^\rho$ is moreover *normal* in the sense that $T^\rho(\mathbf{1})$ is a trivial object of $\mathcal{C}$; see Section 7.

3.2. **The category $\mathcal{C}\rtimes T$.** Let $\mathcal{C}$ be a finite tensor category over $\Bbbk$ and let $T$ be a $\Bbbk$-linear right exact Hopf monad on $\mathcal{C}$ . The category $\mathcal{C}$ is a $\mathcal{C}^T$-module through the tensor functor $\mathcal{F}:\mathcal{C}^T\to\mathcal{C}$. This means that the action $\overline{\otimes}:\mathcal{C}^T\times\mathcal{C}\to\mathcal{C}$ is given by $(X,s)\overline{\otimes}Y=X\otimes Y$, for all $(X,s)\in\mathcal{C}^T$, $Y\in\mathcal{C}$.

Suppose that $T$ is a *faithful* Hopf monad, or equivalently, that $\mathcal{F}$ is a dominant tensor functor [7, Proposition 4.1]. Then $\mathcal{C}$ is an exact indecomposable $\mathcal{C}^T$-module; see Example 2.1.

We shall use the notation $\mathcal{C}\rtimes T$ to indicate the category $\operatorname{End}_{\mathcal{C}^T}(\mathcal{C})$ of $\mathcal{C}^T$-module endofunctors of $\mathcal{C}$.

Observe that since $\mathcal{C}^T$ is a finite tensor category (Lemma 3.5) and $\mathcal{C}$ is an exact $\mathcal{C}^T$-module, then every $\Bbbk$-linear module endofunctor of $\mathcal{C}$ is exact [14, Proposition 3.11]. Moreover, it follows from [14, Proposition 3.23] that the category $\mathcal{C} \rtimes T$ is again a finite tensor category over $\Bbbk$.

By [14, Lemma 3.25], $\mathcal{C}$ is an exact indecomposable $\mathcal{C} \rtimes T$-module with respect to the action $\overline{\otimes} : (\mathcal{C} \rtimes T) \times \mathcal{C} \to \mathcal{C}$ given by

$$F \overline{\otimes} X = F(X),$$

for all $F \in \mathcal{C} \rtimes T, X \in \mathcal{C}$.

The third part of the next lemma is a particular case of [14, Theorem 3.27]. We shall include the proof for the reader's convenience.

**Lemma 3.7.** *Let $\mathcal{C}$ be a finite tensor category and let $T$ be a right exact faithful Hopf monad on $\mathcal{C}$. Then the following hold:*

1. *The functor $R : \mathcal{C}^{\mathrm{rev}} \to \mathcal{C} \rtimes T$, $R(X) = R_X$, where for any $X \in \mathcal{C}$, $R_X : \mathcal{C} \to \mathcal{C}$ is given by $R_X(Y) = Y \otimes X$, is a full embedding of tensor categories.*
2. *Let, for all $(X, s) \in \mathcal{C}^T$, $Y \in \mathcal{C}$,*
$$b_{X,Y} : T(X \otimes Y) \xrightarrow{\xi_{X,Y}} T(X) \otimes T(Y) \xrightarrow{s \otimes \mathrm{id}} X \otimes T(Y).$$
*Then $(T, b)$ is an algebra in $\mathcal{C} \rtimes T$, with multiplication $\mu : T^2 \to T$ and unit $\eta : \mathrm{id}_{\mathcal{C}} \to T$.*
3. *There is an equivalence of tensor categories $\mathcal{C}^T \simeq \mathrm{End}_{\mathcal{C} \rtimes T}(\mathcal{C})$.*

*Proof.* 1. It is not difficult to see that for any $X \in \mathcal{C}$ the functor $R_X : \mathcal{C} \to \mathcal{C}$ is a $\mathcal{C}^T$-module functor and $R_{X \otimes Y} = R_Y \circ R_X$.

2. It follows from Remark 3.3 that the composition

$$b_{X,Y} : T(X \otimes Y) \xrightarrow{\xi_{X,Y}} T(X) \otimes T(Y) \xrightarrow{s \otimes \mathrm{id}} X \otimes T(Y)$$

is an isomorphism for any $(X, s) \in \mathcal{C}^T$, $Y \in \mathcal{C}$.

To show that $(T, b)$ is a module functor we have to prove that equations (2.3), (2.4) are fulfilled. Let $(X, s), (Y, r) \in \mathcal{C}^T$ and $Z \in \mathcal{C}$. In this case the left hand side of (2.3) equals

$$\begin{aligned}
(\mathrm{id}_X \overline{\otimes} b_{Y,Z}) b_{X, Y \otimes Z} &= (\mathrm{id}_X \otimes (r \otimes \mathrm{id}_{T(Z)}) \xi_{Y,Z})(s \otimes \mathrm{id}_{T(Y \otimes Z)}) \xi_{X, Y \otimes Z} \\
&= (s \otimes r \otimes \mathrm{id}_{T(Z)})(\mathrm{id}_X \otimes \xi_{Y,Z}) \xi_{X, Y \otimes Z} \\
&= (s \otimes r \otimes \mathrm{id}_{T(Z)})(\xi_{X,Y} \otimes \mathrm{id}_{T(Z)}) \xi_{X \otimes Y, Z} \\
&= b_{X \otimes Y, Z}.
\end{aligned}$$

The third equality follows from (3.3). This proves equation (2.3). Equation (2.4) follows similarly, using (3.4). Hence $(T, b) \in \mathcal{C} \rtimes T$.

In the same fashion, now using the relations (3.2), (3.6) and the naturality of $\xi$, it is shown that $\mu : (T, b) \otimes (T, b) \to (T, b)$ and $\eta : (\mathrm{id}_{\mathcal{C}}, \mathrm{id}) \to (T, b)$ satisfy condition (2.6), that is, they are morphisms of $\mathcal{C}^T$-module functors. This implies that $(T, b)$ is an algebra in $\mathcal{C} \rtimes T$ as claimed.

3. Define $\Phi : \mathcal{C}^T \to \mathrm{End}_{\mathcal{C}\rtimes T}(\mathcal{C})$ by $\Phi(X,s)(Y) = X\otimes Y$ for all $(X,s) \in \mathcal{C}^T, Y \in \mathcal{C}$. The functor $\Phi(X,s)$ is indeed a $\mathcal{C}\rtimes T$-module functor as follows. If $(F,d^F) \in \mathcal{C}\rtimes T$, $Y \in \mathcal{C}$ define

$$c^{(X,s)}_{F,Y} : \Phi(X,s)(F\overline{\otimes}Y) \to F\overline{\otimes}\Phi(X,s)(Y), \quad c^{(X,s)}_{F,Y} = \left(d^F_{X,Y}\right)^{-1}.$$

Equation (2.3) amounts to

$$F(c^{(X,s)}_{G,Y})c^{(X,s)}_{F,G(Y)} = c^{(X,s)}_{F\circ G,Y}.$$

This last equation is equivalent to equation (2.5). Define $\Psi : \mathrm{End}_{\mathcal{C}\rtimes T}(\mathcal{C}) \to \mathcal{C}^T$ by $\Psi(F,c^F) = (F(\mathbf{1}), s^F)$, where

$$s^F = F(\phi) \circ \left(c^F_{T,\mathbf{1}}\right)^{-1}.$$

It is straightforward to see that that $\Phi$ and $\Psi$ are tensor functors and thus they give the desired equivalence of tensor categories. □

**Example 3.8.** Let $\mathcal{C}$ be a finite tensor category and let $T$ be a right exact faithful Hopf monad on $\mathcal{C}$. Suppose that the forgetful functor $\mathcal{F} : \mathcal{C}^T \to \mathcal{C}$ is perfect, that is, it has an exact right adjoint. Let $(A,\sigma) \in \mathcal{Z}(\mathcal{C}^T)$ be the induced central algebra of $T$.

As explained in Subsection 1.2, the category $(\mathcal{C}^T)_A$ of right $A$-modules in $\mathcal{C}^T$ is a tensor category and there is an equivalence of tensor categories $K : (\mathcal{C}^T)_A \to \mathcal{C}$ such that $KF_A = \mathcal{F}$, where $F_A : \mathcal{C}^T \to (\mathcal{C}^T)_A$ is the free module functor.

Let ${}_A(\mathcal{C}^T)_A$ be the category of $A$-bimodules in $\mathcal{C}^T$; ${}_A(\mathcal{C}^T)_A$ is a tensor category with tensor product $\otimes_A$ and unit object $A$. Observe that the functor $K$ induces an equivalence of $\mathcal{C}^T$-module categories $(\mathcal{C}^T)_A \simeq \mathcal{C}$. We thus obtain an equivalence of tensor categories $\mathcal{C}\rtimes T \simeq ({}_A(\mathcal{C}^T)_A)^{\mathrm{rev}}$; see [14, Proof of Lemma 3.25].

Under this equivalence, the full embedding in Lemma 3.7 (1) corresponds to the full embedding $(\mathcal{C}^T)_A \to {}_A(\mathcal{C}^T)_A$ induced by the half-braiding $\sigma$.

## 4. Module categories over $\mathcal{C}^T$

Along this section $\mathcal{C}$ will be a monoidal category. For any Hopf monad $T$ on $\mathcal{C}$ we shall give a construction of module categories over $\mathcal{C}^T$ from module categories over $\mathcal{C}$. The monad structure is denoted by $(T,\mu,\eta)$ and the comonoidal structure by $(T,\xi,\phi)$.

4.1. **$T$-equivariant module categories.** Let $T$ be a Hopf monad over the monoidal category $\mathcal{C}$ and let $\mathcal{M}$ be a $\mathcal{C}$-module.

The functor $T$, being comonoidal, induces a structure of lax $\mathcal{C}$-module category on $\mathcal{M}$, denoted $\mathcal{M}(T)$. The action on $\mathcal{M}(T)$ is given in the form $X\overline{\otimes}^T M = T(X)\overline{\otimes}M$, and the associativity and unit morphisms are defined by

$$m^T_{X,Y,M} = m_{T(X),T(Y),M}\,(\xi_{X,Y}\overline{\otimes}\,\mathrm{id}_M), \quad l^T_M = l_M(\phi\overline{\otimes}\mathrm{id}_M),$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$. See Subsection 3.1.

**Lemma 4.1.** *Let $(U,c):\mathcal{M}\to\mathcal{M}(T)$ be a lax $\mathcal{C}$-module functor. Then the pair $(U^2,d):\mathcal{M}\to\mathcal{M}(T)$ is a lax $\mathcal{C}$-module functor, where*

$$d_{X,M}=(\mu_X\overline{\otimes}\mathrm{id}_{U^2(M)})c_{T(X),U(M)}U(c_{X,M}),\tag{4.1}$$

*for all $X\in\mathcal{C}$, $M\in\mathcal{M}$.*

*Proof.* For every $M\in\mathcal{M}$ we have

$$\begin{aligned}
l^T_{U^2(M)}d_{\mathbf{1},\mathbf{M}}&=l_{U^2(M)}(\phi\otimes\mathrm{id}_{U(M)})d_{\mathbf{1},\mathbf{M}}\\
&=l_{U^2(M)}(\phi\mu_{\mathbf{1}}\overline{\otimes}\mathrm{id}_{U^2(M)})c_{T(\mathbf{1}),U(M)}U(c_{\mathbf{1},M})\\
&=l_{U^2(M)}(\phi T(\phi)\overline{\otimes}\mathrm{id}_{U^2(M)})c_{T(\mathbf{1}),U(M)}U(c_{\mathbf{1},M})\\
&=l_{U^2(M)}(\phi\overline{\otimes}\mathrm{id}_{U^2(M)})c_{\mathbf{1},U(M)}U(\phi\overline{\otimes}\mathrm{id}_{U(M)})U(c_{\mathbf{1},M})\\
&=U(l_{U(M)})U((\phi\overline{\otimes}\mathrm{id}_{U(M)})c_{\mathbf{1},M})\\
&=U^2(l_M),
\end{aligned}$$

the second equality by (3.6), the third equality by the naturality of $c$, and the fifth because $c$ is a lax module functor. Hence $d$ satisfies (2.4). Now the left hand side of equation (2.3) equals

$$\begin{aligned}
&(\mu_X\overline{\otimes}(\mu_Y\overline{\otimes}\mathrm{id}_{U^2(M)}))(\mathrm{id}_{T^2(X)}\overline{\otimes}c_{T(Y),U(M)}U(c_{Y,M}))\\
&\qquad\qquad\times c_{T(X),U(T(Y)\overline{\otimes}M))}U(c_{X,Y\overline{\otimes}M})U^2(c_{m_{X,Y,M}}).
\end{aligned}$$

On the other hand, the right hand side of (2.3) equals

$$\begin{aligned}
&m_{T(X),T(Y),U^2(M)}(\xi_{X,Y}\mu_{X\otimes Y}\overline{\otimes}\mathrm{id}_{U^2(M)})c_{T(X\otimes Y),U(M)}U(c_{X\otimes Y,M})\\
&=m_{T(X),T(Y),U^2(M)}((\mu_X\otimes\mu_Y)\xi_{T(X),T(Y)}T(\xi_{X,Y})\overline{\otimes}\mathrm{id}_{U^2(M)})\\
&\quad\times c_{T(X\otimes Y),U(M)}U(c_{X\otimes Y,M})\\
&=m_{T(X),T(Y),U^2(M)}((\mu_X\otimes\mu_Y)\xi_{T(X),T(Y)}\overline{\otimes}\mathrm{id}_{U^2(M)})\\
&\quad\times c_{T(X)\otimes T(Y),M}U(\xi_{X,Y}\otimes\mathrm{id}_M)U(c_{X\otimes Y,M})\\
&=(\mu_X\overline{\otimes}(\mu_Y\overline{\otimes}\mathrm{id}_{U^2(M)}))m_{T^2(X),T^2(Y),U^2(M)}(\xi_{T(X),T(Y)}\overline{\otimes}\mathrm{id}_{U^2(M)})\\
&\quad\times c_{T(X)\otimes T(Y),M}U(\xi_{X,Y}\overline{\otimes}\mathrm{id}_M)U(c_{X\otimes Y,M})\\
&=(\mu_X\overline{\otimes}(\mu_Y\overline{\otimes}\mathrm{id}_{U^2(M)}))(\mathrm{id}_{T^2(X)}\overline{\otimes}c_{T(Y),U(M)})c_{T(X),T(Y)\overline{\otimes}U(M)}\\
&\quad\times U(m_{T(X),T(Y),U(M)}(\xi_{X,Y}\overline{\otimes}\mathrm{id}_M)c_{X\otimes Y,M})\\
&=(\mu_X\overline{\otimes}(\mu_Y\overline{\otimes}\mathrm{id}_{U^2(M)}))(\mathrm{id}_{T^2(X)}\overline{\otimes}c_{T(Y),U(M)})c_{T(X),T(Y)\overline{\otimes}U(M)}\\
&\quad\times U((\mathrm{id}_X\overline{\otimes}c_{Y,M})c_{X,Y\overline{\otimes}M})U^2(c_{m_{X,Y,M}})
\end{aligned}$$

The second equality follows from (3.5), the third equality follows from the naturality of $c$, the fourth by the naturality of $m$, the fifth and sixth equalities follow because $(U,c)$ is a lax module functor. □

**Definition 4.2.** A *$T$-equivariant $\mathcal{C}$-module category* is a triple $(\mathcal{M},U,c)$, where:

- $\mathcal{M}$ is a $\mathcal{C}$-module category,
- $(U,c):\mathcal{M}\to\mathcal{M}(T)$ is a lax $\mathcal{C}$-module functor,

- $(U, \nu, u)$ is a monad on $\mathcal{M}$ and both natural transformations $\nu : U^2 \to U$ and $u : \mathrm{Id}_{\mathcal{M}} \to U$ are morphisms of $\mathcal{C}$-module functors.

*Remark* 4.3. The module functor structures of $\mathrm{Id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}(T)$ and $U^2 : \mathcal{M} \to \mathcal{M}(T)$, implicit in the third condition of the definition, are the ones given by $\eta\overline{\otimes}\mathrm{id}$ and Lemma 4.1, respectively.

Let $(\mathcal{M}, U, c)$ be a $T$-equivariant $\mathcal{C}$-module. Since $U$ is a monad on $\mathcal{M}$, we may consider the category $\mathcal{M}^U$ of $U$-modules in $\mathcal{M}$. The objects $(M, s) \in \mathcal{M}^U$ will be called *$U$-equivariant objects*.

**Example 4.4.** The monoidal category $\mathcal{C}$ is a module category over itself and $(T, \xi) : \mathcal{C} \to \mathcal{C}(T)$ is a lax $\mathcal{C}$-module functor. It follows from (3.5) that $\mu : (T^2, d) \to (T, \xi)$ is a natural transformation of module functors, hence $\mathcal{C}$ is a $T$-equivariant $\mathcal{C}$-module category.

**Example 4.5.** (Module categories over equivariantizations.) Let $G$ be a finite group and $\rho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$ be an action of $G$ on $\mathcal{C}$. As explained in Example 3.6, the endofunctor $T^\rho = \oplus_{\sigma\in G}\, \rho^\sigma$ has a structure of Hopf monad over $\mathcal{C}$ such that the equivariantization $\mathcal{C}^G$ is tensor equivalent to $\mathcal{C}^{T^\rho}$.

Let $F \subseteq G$ be a subgroup. Recall that an $F$-equivariant $\mathcal{C}$-module [13] is a module category $\mathcal{M}$ over $\mathcal{C}$ endowed with a family of module functors $(U_\sigma, c^\sigma) : \mathcal{M} \to \mathcal{M}(\rho^\sigma)$ for any $\sigma \in F$ and a family of natural isomorphisms $\mu_{\sigma,\tau} : (U_\sigma \circ U_\tau, b) \to (U_{\sigma\tau}, c^{\sigma\tau})$ $\sigma, \tau \in F$ such that

$$(\mu_{\sigma,\tau\nu})_M \circ U_\sigma(\mu_{\tau,\nu})_M = (\mu_{\sigma\tau,\nu})_M \circ (\mu_{\sigma,\tau})_{U_\nu(M)}, \tag{4.2}$$

$$c^{\sigma\tau}_{X,M} \circ (\mu_{\sigma,\tau})_{X\overline{\otimes}M} = ((\gamma_{\sigma,\tau})_X \overline{\otimes} (\mu_{\sigma,\tau})_M) \circ c^{\sigma}_{\rho^\tau(X),U_\tau(M)} \circ U_\sigma(c^\tau_{X,M}), \tag{4.3}$$

for all $\sigma, \tau, \nu \in F$, $X \in \mathcal{C}$, $M \in \mathcal{M}$. The functor $U : \mathcal{M} \to \mathcal{M}(T^\rho)$, $U = \oplus_{\sigma\in F} U_\sigma$ makes the category $\mathcal{M}$ $T^\rho$-equivariant. The category of $U$-equivariant objects in $\mathcal{M}$ coincides with the category of $F$-equivariant objects in $\mathcal{M}$ in the sense of [13].

**Example 4.6.** (Module categories over $H$-mod.) Consider the tensor category $\mathcal{C} = \mathrm{vect}_{\Bbbk}$ of finite dimensional $\Bbbk$-vector spaces. Let $H$ be a finite dimensional Hopf algebra over $\Bbbk$. Then $H \otimes - : \mathcal{C} \to \mathcal{C}$ is a $\Bbbk$-linear right exact Hopf monad on $\mathcal{C}$ and there is an equivalence of tensor categories $\mathcal{C}^{H\otimes -} \simeq H$-mod, where $H$-mod denotes the category of finite dimensional representations of $H$. Moreover, this assignment defines an equivalence between finite dimensional Hopf algebras over $\Bbbk$ and $\Bbbk$-linear Hopf monads on $\mathcal{C}$ [7, Lemma 2.5].

Let $\mathcal{M} = \mathcal{C} = \mathrm{vect}_{\Bbbk}$ denote the canonical $\mathcal{C}$-module category (which is, up to equivalence, the only indecomposable $\mathcal{C}$-module category). Let also $T = H\otimes -$ the Hopf monad associated to the finite dimensional Hopf algebra $H$. Then $T$-equivariant $\mathcal{C}$-module category structures on $\mathcal{M}$ correspond to finite dimensional left $H$-comodule algebras, as the next lemma shows.

**Lemma 4.7.** *Let $\mathcal{C}$, $T$ and $\mathcal{M}$ as in Example 4.6. Then there is an equivalence between the categories of $T$-equivariant $\mathcal{C}$-module category structures on $\mathcal{M}$ and finite dimensional left $H$-comodule algebras.*

*Proof.* Every finite dimensional $\Bbbk$-algebra $A$ induces canonically a $\Bbbk$-linear monad $U = A \otimes_{\Bbbk} -$ on $\text{vect}_{\Bbbk}$. Conversely, every such monad $U$ is isomorphic to the one induced by a finite dimensional $\Bbbk$-algebra $A$: indeed, as in the proof of [7, Lemma 2.5], we get that $U \simeq U(\mathbf{1}) \otimes -$ as $\Bbbk$-linear functors, and since $U$ is a monad then $A = U(\mathbf{1})$ is an algebra [9, Example 1.2].

Under this correspondence, the conditions on $U = A \otimes -$ in Definition 4.2 correspond to the condition that $A$ is a left $H$-comodule algebra. Indeed, a structure $c$ of lax $\mathcal{C}$-module functor on $U$ is uniquely determined by a map $c : A \to A \otimes H$ making $A$ into a right $H$-comodule, in view of (2.3) and (2.4). The requirement that the multiplication and unit of $U = A \otimes -$ are morphisms of module functors amounts to the condition that the multiplication and unit of $A$ are comodule maps. Thus $A$ is a right $H$-comodule algebra, as claimed. □

**Definition 4.8.** Let $(\mathcal{M}, U, c)$, $(\mathcal{M}, \widetilde{U}, \widetilde{c})$ be two $T$-equivariant structures on the module category $\mathcal{M}$.

- A *morphism of $T$-equivariant structures* $\alpha : (\mathcal{M}, U, c) \to (\mathcal{M}, \widetilde{U}, \widetilde{c})$ is a monad morphism $\alpha : U \to \widetilde{U}$ such that $\alpha$ is also a morphism of lax module functors.
- A morphism of $T$-equivariant structures $\alpha : (\mathcal{M}, U, c) \to (\mathcal{M}, \widetilde{U}, \widetilde{c})$ is *surjective* if $\alpha_M$ is an epimorphism for any $M \in \mathcal{M}$.
- We say that a $T$-equivariant structure $(\mathcal{M}, U, c)$ is *simple* if any surjective morphism of $T$-equivariant structures $\alpha : U \to \widetilde{U}$ is an isomorphism.

**Theorem 4.9.** *Let $\mathcal{M}$ be a $T$-equivariant $\mathcal{C}$-module with lax module functor $(U, c) : \mathcal{M} \to \mathcal{M}(T)$. Then the following hold.*

1. *The category $\mathcal{M}^U$ has a structure of $\mathcal{C}^T$-module category.*
2. *If $\alpha : (\mathcal{M}, U, c) \to (\mathcal{M}, \widetilde{U}, \widetilde{c})$ is a morphism of $T$-equivariant structures, the functor $\alpha^* : \mathcal{M}^{\widetilde{U}} \to \mathcal{M}^U$, $\alpha^*(M, s) = (M, s \circ \alpha_M)$ for all $(M, s) \in \mathcal{M}^{\widetilde{U}}$, is a $\mathcal{C}^T$-module functor.*
3. *If $\mathcal{M}^U$ is a simple $\mathcal{C}^T$-module category then $(\mathcal{M}, U, c)$ is simple.*

*Proof.* 1. Let us define the action $\overline{\otimes} : \mathcal{C}^T \times \mathcal{M}^U \to \mathcal{M}^U$ as follows. Let $(X, r)$ be an object in $\mathcal{C}^T$ and $(M, s)$ be a $U$-equivariant object in $\mathcal{M}$, then $(X, r)\overline{\otimes}(M, s) = (X\overline{\otimes}M, (r\overline{\otimes}s)c_{X,M})$. Let us prove that the object $(X\overline{\otimes}M, (r\overline{\otimes}s)c_{X,M})$ is $U$-equivariant. For this, we need to show that the map $t = (r\overline{\otimes}s)c_{X,M}$ satisfies $t \circ U(t) = t \circ \nu_{X\overline{\otimes}M}$. Indeed:

$$\begin{aligned} t \circ U(t) &= (r\overline{\otimes}s)c_{X,M}U(r\overline{\otimes}s)U(c_{X,M}) \\ &= (rT(r)\overline{\otimes}sU(s))c_{T(X),U(M)}U(c_{X,M}) \\ &= (r\mu_X\overline{\otimes}s\nu_M)c_{T(X),U(M)}U(c_{X,M}) \\ &= (r\overline{\otimes}s\nu_M)d_{X,M} = (r\overline{\otimes}s)c_{X,M}\nu_{X\overline{\otimes}M} = t \circ \nu_{X\overline{\otimes}M}. \end{aligned}$$

The first equality follows from the naturality of $c$, the second follows because $(X, r)$ is a $T$-module and $(M, s)$ is $U$-equivariant. The third equality follows from the definition of $d$, see equation (4.1), the last equality follows since $\nu$ is a module functor. The associativity and unit isomorphisms are the obvious ones.

2. Since $\alpha : U \to \widetilde{U}$ is a morphism of monads, it follows from [9, Lemma 1.7] that $\alpha$ induces a functor $\alpha^* : \mathcal{M}^{\widetilde{U}} \to \mathcal{M}^U$ such that $\mathcal{F}_U\alpha^* = \mathcal{F}_{\widetilde{U}}$.

Let us show that $\alpha^*$ is a $\mathcal{C}^T$-module functor. Let be $(X, r) \in \mathcal{C}^T$, $(M, s) \in \mathcal{M}^U$. Then

$$\alpha^*((X, r)\overline{\otimes}(M, s)) = (X\overline{\otimes}M, (r\overline{\otimes}s)c_{X,M}\alpha_{X\overline{\otimes}M}),$$

and

$$(X, r)\overline{\otimes}\alpha^*(M, s) = (X\overline{\otimes}M, (r\overline{\otimes}s\alpha_M)\widetilde{c}_{X,M}).$$

Since $\alpha : U \to \widetilde{U}$ is a morphism of lax module functors, it follows from equation (2.6) that $\alpha^*((X, r)\overline{\otimes}(M, s)) = (X, r)\overline{\otimes}\alpha^*(M, s)$.

3. Assume $(\mathcal{M}, \widetilde{U}, \widetilde{c})$ is another $T$-equivariant structure and $\alpha : U \to \widetilde{U}$ is a surjective morphism of $T$-equivariant structures. Notice that $\alpha^*(f) = f$ for all morphism $f$ in $\mathcal{M}^{\widetilde{U}}$. Hence $\alpha^*$ is a faithful functor.

The functor $\alpha^*$ is also full. Indeed, let be $(M, s), (N, r) \in \mathcal{M}^{\widetilde{U}}$ and $f : (M, s\alpha_M) \to (N, r\alpha_N)$ be a morphism in $\mathcal{M}^U$. Then $fs\alpha_M = r\alpha_N U(f)$, which implies, by the naturality of $\alpha$, that $fs\alpha_M = r\widetilde{U}(f)\alpha_M$. Since $\alpha_M$ is surjective, then $fs = r\widetilde{U}(f)$ and $f$ is a morphism in $\mathcal{M}^{\widetilde{U}}$.

Since $\mathcal{M}^U$ is a simple $\mathcal{C}^T$-module category then the functor $\alpha^*$ is an equivalence of module categories. Hence there exists a module functor $\mathcal{F} : \mathcal{M}^U \to \mathcal{M}^{\widetilde{U}}$ such that $\mathcal{F} \circ \alpha^* \simeq \mathrm{Id}$ and $\alpha^* \circ \mathcal{F} \simeq \mathrm{Id}$. In particular, there exists a natural isomorphism $\gamma : \alpha^* \circ \mathcal{F} \to \mathrm{Id}$. Since $\gamma_{(M,s)}$ is a morphism in the category $\mathcal{M}^U$, for all $(M, s) \in \mathcal{M}^U$, the diagram

$$\begin{array}{ccc} U(\mathcal{F}(M)) & \xrightarrow{U(\gamma)} & U(M) \\ \Big\downarrow{\scriptstyle s^{\mathcal{F}}\alpha_{\mathcal{F}(M)}} & & \Big\downarrow{\scriptstyle s} \\ \mathcal{F}(M) & \xrightarrow{\gamma} & M \end{array} \tag{4.4}$$

is commutative. Here $\mathcal{F}(M,s) = (\mathcal{F}(M), s^{\mathcal{F}})$. Let us define a new functor $\widehat{\mathcal{F}} : \mathcal{M}^U \to \mathcal{M}^{\widetilde{U}}$ as follows. For any $(M,s) \in \mathcal{M}^U$

$$\widehat{\mathcal{F}}(M,s) = (M, \gamma_M \circ s^{\mathcal{F}} \circ \widetilde{U}(\gamma_M^{-1})).$$

For any morphism $f : (M,s) \to (N,r)$ in $\mathcal{M}^U$, $\widehat{\mathcal{F}}(f) = f$.

The object $\widehat{\mathcal{F}}(M,s)$ is in $\mathcal{M}^{\widetilde{U}}$. Indeed, denote $t = \gamma_M \circ s^{\mathcal{F}} \circ \widetilde{U}(\gamma_M^{-1})$, then

$$\begin{aligned} t\widetilde{U}(t) &= \gamma_M s^{\mathcal{F}} \widetilde{U}(\gamma_M^{-1}) \widetilde{U}(\gamma_M s^{\mathcal{F}} \widetilde{U}(\gamma_M^{-1})) = \gamma_M s^{\mathcal{F}} \widetilde{U}(s^{\mathcal{F}}) \widetilde{U}^2(\gamma_M^{-1}) \\ &= \gamma_M s^{\mathcal{F}} \widetilde{\nu}_{\mathcal{F}(M)} \widetilde{U}^2(\gamma_M^{-1}) = \gamma_M s^{\mathcal{F}} \widetilde{U}(\gamma_M^{-1}) \widetilde{\nu}_M = t\widetilde{\nu}_M. \end{aligned}$$

The third equality follows since $(\mathcal{F}(M), s^{\mathcal{F}}) \in \mathcal{M}^{\widetilde{U}}$ and the fourth equality follows from the naturality of $\widetilde{\nu}$. Now, using [9, Lemma 1.6], it follows that there exists a monad morphism $\beta : U \to \widetilde{U}$ such that $\widehat{\mathcal{F}} = \beta^*$. Whence, for any $(M,s) \in \mathcal{M}^U$ we have $\gamma_M s^{\mathcal{F}} = s\beta_M \widetilde{U}(\gamma_M)$. Hence

$$\begin{aligned} \gamma_M s^{\mathcal{F}} \alpha_{\mathcal{F}(M)} &= s\beta_M \widetilde{U}(\gamma_M) \alpha_{\mathcal{F}(M)} \\ &= s\beta_M \alpha_M U(\gamma_M). \end{aligned}$$

Using commutativity of diagram (4.4), we obtain that $s\beta_M\alpha_M = s$. Since this argument can be applied to $(U(M), \nu_M)$ for any $M \in \mathcal{M}$, we get that $\nu_M \beta_{U(M)} \alpha_{U(M)} = \nu_M$. Thus

$$\mathrm{id}_M = \nu_M U(u_M) = \nu_M \beta_{U(M)} \alpha_{U(M)} U(u_M) = \beta_M \alpha_M.$$

An analogous argument shows that $\alpha_M \beta_M = \mathrm{id}_M$, and therefore $\alpha$ is an isomorphism. Thus we conclude that the equivariant structure $(\mathcal{M}, U, c)$ is simple. ☐

**Proposition 4.10.** *Let $\mathcal{M}, \widetilde{\mathcal{M}}$ be $\mathcal{C}$-module categories and assume that $(\mathcal{M}, U, c)$, $(\widetilde{\mathcal{M}}, \widetilde{U}, \widetilde{c})$ are $T$-equivariant structures. Suppose that $(G, d) : \mathcal{M} \to \widetilde{\mathcal{M}}$ is a $\mathcal{C}$-module functor. Let $\widehat{T}(G) : \mathcal{M}(T) \to \widetilde{\mathcal{M}}(T)$ be the $\mathcal{C}$-module functor defined, for each $M \in \mathcal{M}(T)$, by $\widehat{T}(G)(M) = G(M)$, with module structure*

$$d_{T(X),M} : G(T(X)\overline{\otimes} M) \to T(X)\overline{\otimes} G(M).$$

*Then the following assertions hold.*

1. *Suppose there exists a $\mathcal{C}$-module natural transformation $\theta : \widetilde{U} \circ G \to \widehat{T}(G) \circ U$ such that*

$$\theta_M \widetilde{\nu}_{G(M)} = G(\nu_M) \theta_{U(M)} \widetilde{U}(\theta_M), \quad \theta_M \widetilde{u}_{G(M)} = G(u_M), \tag{4.5}$$

   *for all $M \in \mathcal{M}$, where the functor $\widehat{G} : \mathcal{M}^U \to \widetilde{\mathcal{M}}^{\widetilde{U}}$ is defined by $\widehat{G}(M,s) = (G(M), G(s)\theta_M)$, for all $(M,s) \in \mathcal{M}^U$.*

   *Then the functor $\widehat{G}$ is a $\mathcal{C}^T$-module functor.*
2. *Assume that $(H,h) : \mathcal{M} \to \widetilde{\mathcal{M}}$ is another $\mathcal{C}$-module functor equipped with a $\mathcal{C}$-module natural transformation $\chi : \widetilde{U} \circ H \to \widehat{T}(H) \circ U$*

*satisfying* (4.5) *(replacing $G$ by $H$) and let $\alpha : (G, d) \to (H, h)$ be a $\mathcal{C}$-module natural transformation such that*

$$\alpha_{U(M)}\theta_M = \chi_M \widetilde{U}(\alpha_M), \tag{4.6}$$

*for all $M \in \mathcal{M}$. Then, the natural transformation $\widehat{\alpha} : \widehat{G} \to \widehat{H}$, $\widehat{\alpha}_{(M,s)} = \alpha_M$, $(M, s) \in \mathcal{M}^U$, is a $\mathcal{C}^T$-module natural transformation.*

*Proof.* 1. That the functor $\widehat{G}$ is well-defined, that is, $\widehat{G}(M, s) \in \widetilde{\mathcal{M}}^{\widetilde{U}}$ for any $(M, s) \in \mathcal{M}^U$, is a consequence of (4.5). The module structure of the functor $\widehat{G}$ is $d$, the same module structure of the functor $G$. This map is a morphism in the category $\widetilde{\mathcal{M}}^{\widetilde{U}}$ since $\theta$ is a $\mathcal{C}$-module natural transformation.

2. For any $(M, s) \in \mathcal{M}^U$ the map $\alpha_M$ is a morphism in the category $\widetilde{\mathcal{M}}^{\widetilde{U}}$ since it satisfies (4.6). □

4.2. **Module categories over $\mathcal{C} \rtimes T$.** Let $(T, \mu, \eta)$ be a Hopf monad over $\mathcal{C}$. Then $\overline{T}$ is a Hopf monad over $\mathcal{C}^{\text{rev}}$. Let $\mathcal{N}$ be a left $\mathcal{C}^{\text{rev}} \rtimes \overline{T}$-module. It follows from Lemma 3.7 (1) that $\mathcal{N}$ is a left $\mathcal{C}$-module. The left action is given by

$$\odot : \mathcal{C} \times \mathcal{N} \to \mathcal{N}, \quad X \odot N = R_X \overline{\otimes} N,$$

for all $X \in \mathcal{C}$, $N \in \mathcal{N}$, and the associativity

$$m_{X,Y,N} : (X \otimes Y) \odot N \to X \odot (Y \odot N), m_{X,Y,N} = m_{R_X, R_Y, N},$$

for all $X, Y \in \mathcal{C}$, $N \in \mathcal{N}$. If $f : X \to Y$ is a morphism in $\mathcal{C}$ and $g : N \to M$ is a morphism in $\mathcal{N}$ then $f \odot g = \alpha_f \overline{\otimes} g$, where $\alpha_f : R_X \to R_Y$ is the natural transformation $(\alpha_f)_Z : X \otimes Z \to Y \otimes Z, (\alpha_f)_Z = f \otimes \mathrm{id}_Z$, for all $Z \in \mathcal{C}$.

Assume that $T$ is right exact and faithful. By Lemma 3.7 (2), $(\overline{T}, b) \in \mathcal{C}^{\text{rev}} \rtimes \overline{T}$. Let $U : \mathcal{N} \to \mathcal{N}$ be the functor defined by $U(N) = \overline{T} \overline{\otimes} N$, for all $N \in \mathcal{N}$, and let $\nu : U^2 \to U$ and $u : \mathrm{Id}_{\mathcal{N}} \to U$ be the natural transformations

$$\nu_N = (\mu \overline{\otimes} \mathrm{id}_N) m^{-1}_{T,T,N}, \quad u_N = \eta \overline{\otimes} \mathrm{id}_N,$$

for all $N \in \mathcal{N}$.

**Lemma 4.11.** *$(U, \nu, u)$ is a monad on $\mathcal{N}$.*

*Proof.* Since $\mathcal{N}$ is a left $\mathcal{C}^{\text{rev}} \rtimes \overline{T}$-module, then there is a monoidal functor $L : \mathcal{C}^{\text{rev}} \rtimes \overline{T} \to \mathrm{End}(\mathcal{N})$, defined in the form $L(X) = X \overline{\otimes} N$, $N \in \mathcal{N}$, where the monoidal structure $c_{X,Y} : L(X \otimes^{\text{rev}} Y) \to L(X) \otimes L(Y)$ is the natural transformation given by

$$(c_{X,Y})_N = m^{-1}_{X,Y,N},$$

for all $X, Y \in \mathcal{C}^{\text{rev}} \rtimes \overline{T}$, $N \in \mathcal{N}$. In particular, $L$ takes algebras in $\mathcal{C}^{\text{rev}} \rtimes \overline{T}$ to algebras in $\mathrm{End}(\mathcal{N})$, that is, to monads on $\mathcal{N}$. This implies the lemma, in view of Lemma 3.7 (2). □

**Theorem 4.12.** *Let $T$ be a right exact faithful Hopf monad on $\mathcal{C}$. Then the following hold:*

1. *With the above module structure, $\mathcal{N}$ is a $T$-equivariant left $\mathcal{C}$-module.*

2. *There is an equivalence of $\mathcal{C}^T$-module categories*

$$\mathcal{N}^U \simeq \operatorname{Hom}_{\mathcal{C}^{\mathrm{rev}} \rtimes \overline{T}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{N}).$$

*Proof.* 1. We shall prove that the functor $U : \mathcal{N} \to \mathcal{N}$ gives a $T$-equivariant structure on $\mathcal{N}$.

Let us first show that $U : \mathcal{N} \to \mathcal{N}(T)$ is a lax module functor. For any $X \in \mathcal{C}$, let $\xi^X : T \circ R_X \to R_{T(X)} \circ T$ be the natural transformation given by $(\xi^X)_Y = \xi_{X,Y}$, $Y \in \mathcal{C}$. For any $X \in \mathcal{C}, N \in \mathcal{N}$, define $c_{X,N} : U(X \odot N) \to T(X) \odot U(N)$ by

$$c_{X,N} = m_{R_{T(X)},T,N}(\xi^X \overline{\otimes} \mathrm{id}_N) m^{-1}_{T,R_X,N}. \tag{4.7}$$

Let us prove that for all $X, Y \in \mathcal{C}$, $N \in \mathcal{N}$,

$$(\mathrm{id}_X \odot c_{Y,N}) c_{X,Y\odot N} U(m_{R_X,R_Y,N}) = m_{R_X,R_Y,T(N)}(\xi_{X,Y} \odot \mathrm{id}_{T}(N)) c_{X\otimes Y,N}. \tag{4.8}$$

The right hand side of (4.8) equals

$$\begin{aligned} &= m_{R_{T(X)},R_{T(Y)},T\overline{\otimes}N}(\xi_{X,Y} \odot \mathrm{id}_{T\overline{\otimes}N}) m_{R_{T(X\otimes Y)},T,N}(\xi^{X\otimes Y}\overline{\otimes}\mathrm{id}_N) m^{-1}_{T,R_{X\otimes Y},N} \\ &= m_{R_{T(X)},R_{T(Y)},T\overline{\otimes}N} m_{R_{T(X)\otimes T(Y)},T,N}(\xi_{X,Y} \odot \mathrm{id}_N)(\xi^{X\otimes Y}\overline{\otimes}\mathrm{id}_N) m^{-1}_{T,R_{X\otimes Y},N}. \end{aligned} \tag{4.9}$$

The left hand side of (4.8) equals

$$\begin{aligned}
&= (\mathrm{id}_X \odot m_{R_{T(Y)},T,N}(\xi^Y\overline{\otimes}\mathrm{id}_N) m^{-1}_{T,R_Y,N}) m_{R_{T(X)},T,Y\odot N}(\xi^X\overline{\otimes}\mathrm{id}_{Y\odot N}) \\
&\qquad \times m^{-1}_{T,R_X,Y\odot N}(\mathrm{id}_T\overline{\otimes} m_{R_X,R_Y,N}) \\
&= (\mathrm{id}_X \odot m_{R_{T(Y)},T,N}(\xi^Y\overline{\otimes}\mathrm{id}_N) m^{-1}_{T,R_Y,N}) m_{R_{T(X)},T,Y\odot N}(\xi^X\overline{\otimes}\mathrm{id}_{Y\odot N}) \\
&\qquad \times m^{-1}_{T,R_X,Y\odot N} m_{T,R_X,R_Y\overline{\otimes}N} m_{T\circ R_X,R_Y,N} m^{-1}_{T,R_{X\otimes Y},N} \\
&= (\mathrm{id}_X \odot m_{R_{T(Y)},T,N}(\xi^Y\overline{\otimes}\mathrm{id}_N) m^{-1}_{T,R_Y,N}) m_{R_{T(X)},T,Y\odot N}(\xi^X\overline{\otimes}\mathrm{id}_{Y\odot N}) \\
&\qquad \times m_{T\circ R_X,R_Y,N} m^{-1}_{T,R_{X\otimes Y},N} \\
&= (\mathrm{id}_X \odot m_{R_{T(Y)},T,N}(\xi^Y\overline{\otimes}\mathrm{id}_N) m^{-1}_{T,R_Y,N}) m_{R_{T(X)},T,Y\odot N} m_{R_{T(X)}\circ T,R_Y,N} \\
&\qquad \times ((\xi^X\overline{\otimes}\mathrm{id}_Y)\otimes\mathrm{id}_N) m^{-1}_{T,R_{X\otimes Y},N} \\
&= (\mathrm{id}_X \odot m_{R_{T(Y)},T,N}(\xi^Y\overline{\otimes}\mathrm{id}_N)) m_{R_{T(X)},TR_Y,N}((\xi^X\otimes\mathrm{id}_Y)\overline{\otimes}\mathrm{id}_N) \\
&\qquad \times m^{-1}_{T,R_{X\otimes Y},N} \\
&= (\mathrm{id}_X \odot m_{R_{T(Y)},T,N}) m_{R_{T(X)},R_{T(Y)}T,N}((\mathrm{id}_X\otimes\xi^Y)(\xi^X\otimes\mathrm{id}_Y)\overline{\otimes}\mathrm{id}_N) \\
&\qquad \times m^{-1}_{T,R_{X\otimes Y},N} \\
&= m_{R_{T(X)},R_{T(Y)},T\overline{\otimes}N} m_{R_{T(X)\otimes T(Y)},T,N}((\mathrm{id}_X\otimes\xi^Y)(\xi^X\otimes\mathrm{id}_Y)\overline{\otimes}\mathrm{id}_N) \\
&\qquad \times m^{-1}_{T,R_{X\otimes Y},N}.
\end{aligned}$$

The second and seventh equalities by (2.1). It follows from (3.3) that this last expression equals (4.9). This proves that $U : \mathcal{N} \to \mathcal{N}(T)$ is a lax module functor. Let us prove now that $(\mathcal{N}, U, c)$ is a $T$-equivariant $\mathcal{C}$-module category. We shall show that $\nu : U^2 \to U$ is a $\mathcal{C}$-module transformation. We must prove that

$$c_{X,N}\nu_{X\odot N} = (\mathrm{id}_X \odot \nu_N)d_{X,N}, \tag{4.10}$$

for all $X \in \mathcal{C}$, $N \in \mathcal{N}$. Here $d_{X,N} = (\mu\overline{\otimes}\mathrm{id}_{U^2(N)})c_{T(X),T\overline{\otimes}N}U(c_{X,N})$ and $c_{X,N}$ is defined in (4.7). We have

$$\begin{aligned} c_{X,N}\nu_{X\odot N} &= m_{R_{T(X)},T,N}(\xi^X\overline{\otimes}\mathrm{id}_N)m^{-1}_{T,R_X,N}(\mu\overline{\otimes}\mathrm{id}_{X\odot N})m^{-1}_{T,T,X\odot N} \\ &= m_{R_{T(X)},T,N}(\xi^X\overline{\otimes}\mathrm{id}_N)(\mu\circ R_X\overline{\otimes}\mathrm{id}_N)m^{-1}_{T^2,R_X,N}m^{-1}_{T,T,X\odot N} \\ &= m_{R_{T(X)},T,N}(\xi^X(\mu\circ R_X)\overline{\otimes}\mathrm{id}_N)m^{-1}_{T,T\circ R_X,N}(\mathrm{id}_T\overline{\otimes}m^{-1}_{T,R_X,N}). \end{aligned} \tag{4.11}$$

The second equality follows from the naturality of $m$ and the third equality follows from the associativity of $m$ (2.1). The right hand side of (4.10) equals

$$\begin{aligned} &= (\mathrm{id}_X \odot (\mu\overline{\otimes}\mathrm{id}_N)m^{-1}_{T,T,N})(\mu_X \odot \mathrm{id}_{U^2(N)})m_{R_{T^2(X)},T,T\overline{\otimes}N}(\xi^{T(X)}\overline{\otimes}\mathrm{id}_{T\overline{\otimes}N}) \\ &\qquad \times m^{-1}_{T,R_{T(X)},T\overline{\otimes}N}(\mathrm{id}_T\overline{\otimes}m_{R_{T(X)},T,N})(\mathrm{id}_T\overline{\otimes}(\xi^X\otimes\mathrm{id}_N)m^{-1}_{T,R_X,N}) \\ &= (\mu_X \odot (\mu\overline{\otimes}\mathrm{id}_N))(\mathrm{id}_{R_{T^2(X)}}\overline{\otimes}m^{-1}_{T,T,N})m_{R_{T^2(X)},T,T\overline{\otimes}N}(\xi^{T(X)}\overline{\otimes}\mathrm{id}_{T\overline{\otimes}N}) \\ &\qquad \times m^{-1}_{T,R_{T(X)},T\overline{\otimes}N}(\mathrm{id}_T\overline{\otimes}m_{R_{T(X)},T,N})(\mathrm{id}_T\overline{\otimes}(\xi^X\overline{\otimes}\mathrm{id}_N)m^{-1}_{T,R_X,N}) \\ &= (\mu_X \odot (\mu\otimes\mathrm{id}_N))m_{R_{T^2(X)},T^2,N}m^{-1}_{R_{T^2(X)}\circ T,T,N}(\xi^{T(X)}\otimes\mathrm{id}_{T\overline{\otimes}N}) \\ &\qquad \times m_{T\circ R_{T(X)},T,N}m^{-1}_{T,R_{T(X)}\circ T,N}(\mathrm{id}_T\overline{\otimes}(\xi^X\otimes\mathrm{id}_N)m^{-1}_{T,R_X,N}) \\ &= (\mu_X \odot (\mu\overline{\otimes}\mathrm{id}_N))m_{R_{T^2(X)},T^2,N}m^{-1}_{R_{T^2(X)}\circ T,T,N}(\xi^{T(X)}\overline{\otimes}\mathrm{id}_{T\overline{\otimes}N}) \\ &\qquad \times m_{T\circ R_{T(X)},T,N}((\mathrm{id}_T\overline{\otimes}\xi^X)\otimes\mathrm{id}_N)m^{-1}_{T,T\circ R_X,N}(\mathrm{id}_T\overline{\otimes}m^{-1}_{T,R_X,N}) \\ &= (\mu_X \odot (\mu\otimes\mathrm{id}_N))m_{R_{T^2(X)},T^2,N}((\xi^{T(X)}\otimes\mathrm{id}_T)\overline{\otimes}\mathrm{id}_N)((\mathrm{id}_T\otimes\xi^X)\overline{\otimes}\mathrm{id}_N) \\ &\qquad \times m^{-1}_{T,T\circ R_X,N}(\mathrm{id}_T\overline{\otimes}m^{-1}_{T,R_X,N}) \\ &= m_{R_{T(X)},T,N}((\alpha_{\mu_X}\otimes\mathrm{id}_T)R_{T^2(X)}(\mu)\otimes\mathrm{id}_N)((\xi^{T(X)}\otimes\mathrm{id}_T)\otimes\mathrm{id}_N) \\ &\qquad \times ((\mathrm{id}_T\otimes\xi^X)\otimes\mathrm{id}_N)m^{-1}_{T,T\circ R_X,N}(\mathrm{id}_T\otimes m^{-1}_{T,R_X,N}) \\ &= m_{R_{T(X)},T,N}((\alpha_{\mu_X}\otimes\mathrm{id}_T)R_{T^2(X)}(\mu)(\xi^{T(X)}\circ T)T(\xi^X)\otimes\mathrm{id}_N) \\ &\qquad \times m^{-1}_{T,T\circ R_X,N}(\mathrm{id}_T\otimes m^{-1}_{T,R_X,N}) \\ &= m_{R_{T(X)},T,N}((\alpha_{\mu_X}\otimes\mathrm{id}_T)(\mathrm{id}_{T^2(X)}\otimes\mu)(\xi^{T(X)}\circ T)T(\xi^X)\otimes\mathrm{id}_N) \\ &\qquad \times m^{-1}_{T,T\circ R_X,N}(\mathrm{id}_T\otimes m^{-1}_{T,R_X,N}). \end{aligned}$$

The second equality follows from the naturality of $m$, the third equality from (2.1), the fourth, fifth and sixth equalities again by the naturality of

$m$. It remains to show that $(\alpha_{\mu_X}\otimes \mathrm{id}_T)(\mathrm{id}_{T^2(X)}\otimes\mu)(\xi^{T(X)}\circ T)T(\xi^X) = \xi^X(\mu\circ R_X)$, but this is (3.5).

2. The category $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}},\mathcal{N})$ is a right $\mathrm{End}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}})$-module category via composition of functors. It follows from Lemma 3.7 and Lemma 3.2 that $\mathrm{End}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}}) \simeq (\mathcal{C}^{\mathrm{rev}})^{\overline{T}} \simeq (\mathcal{C}^T)^{\mathrm{rev}}$. Thus $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}},\mathcal{N})$ is a left $\mathcal{C}^T$-module category using these identifications. Define the $\mathcal{C}^T$-module functors

$$\Phi : \mathcal{N}^U \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}},\mathcal{N}), \quad \Psi : \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}},\mathcal{N}) \to \mathcal{N}^U$$

as follows. If $(N,s)\in\mathcal{N}^U$ then $\Phi(N,s)(X) = R_X\overline{\otimes}N$ for all $X\in\mathcal{C}$. The functor $\Phi(N,s)$ is a module functor with structure given by

$$c^{(N,s)}_{F,Y} : \Phi(N,s)(F\overline{\otimes}Y) \to F\overline{\otimes}\Phi(N,s)(Y), \quad c^{(N,s)}_{F,Y} = (c^F_{-,Y}\otimes\mathrm{id}_N)m^{-1}_{F,R_Y,N},$$

for all $(F,c^F)\in\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}$, $Y\in\mathcal{C}$. If $(G,d^G)\in\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}},\mathcal{N})$ define $\Psi(G,d^G) = (G(\mathbf{1}),s^G)$, where $s^G : U(G(\mathbf{1}))\to G(\mathbf{1})$ is defined by $s^G = G(\phi)\circ(d^G_{T,\mathbf{1}})^{-1}$. Both functors $\Phi,\Psi$ are well-defined $\mathcal{C}^T$-module functors and they give an equivalence of module categories. □

**Corollary 4.13.** *Let $T$ be a right exact faithful Hopf monad on $\mathcal{C}$ and let $\mathcal{M}$ be an exact indecomposable $\mathcal{C}^T$-module category. Then there exists a $T$-equivariant indecomposable exact $\mathcal{C}$-module category $\mathcal{N}$ with simple and exact equivariant structure given by $U:\mathcal{N}\to\mathcal{N}$ such that $\mathcal{M}\simeq\mathcal{N}^U$ as $\mathcal{C}^T$-module categories.*

*Proof.* Let $\mathcal{M}$ be an exact indecomposable left $\mathcal{C}^T$-module category. Thus, $\mathcal{M}$ is an exact indecomposable right $(\mathcal{C}^T)^{\mathrm{rev}}$-module category. Then, the category $\mathcal{N} = \mathrm{Hom}_{(\mathcal{C}^T)^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{rev}},\mathcal{M})$ is an exact indecomposable left $\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}$-module category, see [14, Theorem 3.31]. It follows from Theorem 4.12 that $\mathcal{N} = \mathrm{Hom}_{(\mathcal{C}^T)^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{rev}},\mathcal{M})$ is a $T$-equivariant $\mathcal{C}$-module category with equivariant structure $U:\mathcal{N}\to\mathcal{N}$, $U(N) = \overline{T}\overline{\otimes}N$, and there are $\mathcal{C}^T$-module equivalences

$$\mathcal{N}^U \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\rtimes\overline{T}}(\mathcal{C}^{\mathrm{rev}},\mathcal{N}) \simeq \mathcal{M}.$$

Since the functor $\overline{\otimes}$ is biexact the functor $U$ is exact. □

## 5. Module categories over Hopf algebroids

5.1. **Hopf algebroids.** Let us briefly introduce the notion of Hopf algebroid. The reader is refered to [4], [5], [15]. Let $L$, $R$ be algebras over $\Bbbk$.

**Definition 5.1.** A *left bialgebroid* with base $L$ is a collection $(H,s,t,\Delta,\epsilon)$ where $s: L\to H$, $t: L^{\mathrm{op}}\to H$ are algebra maps such that $s(l)t(l) = t(l)s(l)$ making $H$ an $(L,L)$-bimodule:

$$l\cdot x\cdot l' = s(l)t(l')x,$$

for all $l, l' \in L, x \in H$. The remaining data $\Delta : H \to H \otimes_L H$ and $\epsilon : H \to L$ are $k$-linear maps which make the triple $(H, \Delta, \epsilon)$ into a comonoid in $L$-mod-$L$. Moreover, the following identities are required to hold:

$$\Delta(x)(t(l)\otimes 1) = \Delta(x)(1\otimes s(l)), \tag{5.1}$$

$$\Delta(1) = 1\otimes 1, \tag{5.2}$$

$$\Delta(xy) = \Delta(x)\Delta(y), \tag{5.3}$$

$$\epsilon(1) = 1, \quad \epsilon(xs(\epsilon(y))) = \epsilon(xy) = \epsilon(xt(\epsilon(y))), \tag{5.4}$$

for all $x, y \in H, l \in L$. Right bialgebroids are defined in a similar way. See for example [5, Definition 2.2].

**Definition 5.2.** A *Hopf algebroid* is a collection $(H_R, H_L, \mathcal{S})$ where $H_L = (H, s_L, t_L, \Delta_L, \epsilon_L)$ is a left bialgebroid over $L$ and $H_R = (H, s_R, t_R, \Delta_R, \epsilon_R)$ is a right bialgebroid over $R$, $\mathcal{S} : H \to H$ is a linear map, called the *antipode*, such that

$$s_L \circ \epsilon_L \circ t_R = t_R, \qquad t_L \circ \epsilon_L \circ s_R = s_R, \tag{5.5}$$

$$s_R \circ \epsilon_R \circ t_L = t_L, \qquad t_R \circ \epsilon_R \circ s_L = s_L, \tag{5.6}$$

$$(\Delta_L \otimes \mathrm{id}_H)\Delta_R = (\mathrm{id}_H \otimes \Delta_R)\Delta_L, \quad (\Delta_R \otimes \mathrm{id}_H)\Delta_L = (\mathrm{id}_H \otimes \Delta_L)\Delta_R \tag{5.7}$$

(5.8) $\mathcal{S} : H \to H$ is both an $L$ and $R$ bimodule map:

$$\mathcal{S}(t_L(l)ht_L(l')) = s_L(l')\mathcal{S}(h)s_L(l), \quad \mathcal{S}(t_R(r)ht_R(r')) = s_R(r')\mathcal{S}(h)s_R(r),$$

$$m_H \circ (\mathcal{S} \otimes \mathrm{id}_H) \circ \Delta_L = s_R \circ \epsilon_R, \tag{5.9}$$

$$m_H \circ (\mathrm{id}_H \otimes \mathcal{S}) \circ \Delta_R = s_L \circ \epsilon_L. \tag{5.10}$$

Below, we shall sometimes use the abreviated notation $H$ to indicate the Hopf algebroid $(H_R, H_L, \mathcal{S})$.

*Remark* 5.3. If $(H_R, H_L, \mathcal{S})$ is a Hopf algebroid then $R \simeq L^{\mathrm{op}}$.

If $(H_R, H_L, \mathcal{S})$ is a Hopf algebroid then any left $H$-module is an $L$-bimodule with the left and right $L$-actions given by restriction along the maps $s$ and $t$, respectively [23, Theorem 5.1]. The category $H$-mod of finite-dimensional left $H$-modules is a finite tensor category with monoidal product $\otimes_L$ and unit $L$.

5.2. **Hopf monads and Hopf algebroids.** Let $(H_R, H_L, \mathcal{S})$ be a finite-dimensional Hopf algebroid. Associated to this Hopf algebroid, there is a Hopf monad $T_H$ on the category $L$- mod-$L$ of finite dimensional $L$-bimodules [6, Section 7]. Let $L^e = L\otimes_{\Bbbk} L^{\rm op}$. The category $L$- mod-$L \simeq L^e$- mod is a monoidal category with tensor product $\otimes_L$. The functor $T_H : L^e\text{- mod} \to L^e\text{- mod}$, $T_H(V) = H\otimes_{L^e} V$, for all $V \in L^e$- mod is a Hopf monad with structure maps given by

$$\mu_V : H\otimes_{L^e} H\otimes_{L^e} V \to H\otimes_{L^e} V, \quad \mu_V(x\otimes y\otimes v) = xy\otimes v,$$

$$\eta_V : V \to H\otimes_{L^e} V, \quad \eta_V(v) = 1\otimes v,$$

$$\xi_{V,W} : H\otimes_{L^e}(V\otimes_L W) \to (H\otimes_{L^e} V)\otimes_L(H\otimes_{L^e} W),$$

$$\xi_{V,W}(x\otimes v\otimes w) = x_{(1)}\otimes v\otimes x_{(2)}\otimes w,$$

$$\phi : H\otimes_{L^e} L \to L, \quad \phi(x\otimes l) = \epsilon(xs(l)),$$

for all $V, W \in L^e$- mod, $v \in V, w \in W$, $x, y \in H$, $l \in L$. It follows that $T_H$ is a Hopf monad. Furthermore, there is an equivalence of tensor categories $(L\text{- mod-}L)^{T_H} \simeq H$- mod. See [26, Corollary 5.16]. Any finite tensor category is monoidally equivalent to the category of representations of a Hopf algebroid [6, Theorem 7.6].

5.3. **Comodule algebras over Hopf algebroids.** Let $(H_R, H_L, \mathcal{S})$ be a Hopf algebroid. A *left $H$-comodule algebra* is a triple $(K, s_K, \lambda)$, where $s_K : L \to K$ is an algebra map that makes $K$ in to a $(L, L)$-bimodule

$$l \cdot k \cdot l' = s_K(l)ks_K(l'),$$

for all $k \in K, l, l' \in L$. A left $L$-linear map $\lambda : K \to H\otimes_L K$, such that

$$(\epsilon\otimes \mathrm{id}_K)\lambda = \mathrm{id}_K, \quad (\mathrm{id}_H\otimes\lambda)\,\lambda = (\Delta\otimes \mathrm{id}_K)\,\lambda, \tag{5.11}$$

$$\lambda(K) \subseteq H \times_L K = \{x \in H\otimes_L K : \forall\, l \in L, x(t(l)\otimes 1) = x(1\otimes s_K(l))\} \tag{5.12}$$

$$\lambda(1) = 1\otimes 1, \ \ \lambda(x)\lambda(y) = \lambda(xy), \ \text{ for all } x, y \in K. \tag{5.13}$$

Equation (5.13) makes sense in view of axiom (5.12). We shall use Sweedler's notation: $\lambda(k) = k_{(-1)}\otimes k_{(0)}$, for all $k \in K$.

**Definition 5.4.** We say that a left $H$-comodule algebra $(K, s_K, \lambda)$ is *$H$-simple* if it has no non-trivial $H$-costable ideals.

**Example 5.5.** (1) $(H, s, \Delta)$ is a left $H$-comodule algebra.
(2) $(L, s_L, \lambda_L)$ is a left $H$-comodule algebra, where $s_L = \mathrm{id}_L$ and $\lambda_L : L \to H\otimes_L L$ is *trivial*, that is $\lambda_L(l) = s(l)\otimes 1$, for any $l \in L$.

5.4. **Module categories over Hopf algebroids.** Let $(K, s_K, \lambda)$ be a left $H$-comodule algebra. If $M$ is a left $K$-module then $M$ is a left $L$-module via $s_K$. For any $X \in H\text{-}\mathrm{mod}$ the tensor product $X \otimes_L M$ is a left $K$-module with action given by

$$k \cdot (x \otimes m) = k_{(-1)} \cdot x \otimes k_{(0)} \cdot m, \tag{5.14}$$

for all $k \in K, x \in X, m \in M$. As a consequence of (5.12) this action is well-defined.

**Proposition 5.6.** *Suppose $L$ is semisimple. Then the category $K\text{-}\mathrm{mod}$ is an $H\text{-}\mathrm{mod}$-module as follows. The action is given by*

$$\overline{\otimes} : H\text{-}\mathrm{mod} \times K\text{-}\mathrm{mod} \to K\text{-}\mathrm{mod}, \quad X \overline{\otimes} M = X \otimes_L M,$$

*for all $X \in H\text{-}\mathrm{mod}$, $M \in K\text{-}\mathrm{mod}$. The associativity of the module category is the canonical associativity of the tensor product $\otimes_L$. For any $M \in K\text{-}\mathrm{mod}$, the unit isomorphisms $l_M : L \otimes_L M \to M$ are given by the action of $L$ on $M$.*

*Proof.* The proof is straightforward. Note that the assumption that $L$ is semisimple guarantees the exactness of the module category $K\text{-}\mathrm{mod}$. □

Observe that the exactness assumption on the module category is only needed in Theorem 5.8 below, and the assumption in that theorem is that $L$ is simple (hence semisimple).

For any left $H$-comodule algebra $(K, s_K, \lambda)$ we shall introduce a $T_H$-equivariant structure on the module category $L\text{-}\mathrm{mod}$. Define $U_K : L\text{-}\mathrm{mod} \to L\text{-}\mathrm{mod}$, $U_K(V) = K \otimes_L V$, for all $V \in L\text{-}\mathrm{mod}$. For any $X \in H\text{-}\mathrm{mod}, V \in L\text{-}\mathrm{mod}$ define

$$c_{X,V} : K \otimes_L X \otimes_L V \to (H \otimes_{L^e} X) \otimes_L K \otimes_L V,$$
$$c_{X,V}(k \otimes x \otimes v) = k_{(-1)} \otimes x \otimes k_{(0)} \otimes v.$$

Let $\nu : U_K^2 \to U_K$, $u : \mathrm{Id} \to U_K$, be defined as follows. For any $V \in L\text{-}\mathrm{mod}$

$$\nu_V : K \otimes_L K \otimes_L V \to K \otimes_L V, \quad \nu_V(k \otimes k' \otimes v) = kk' \otimes v,$$
$$u_V : V \to K \otimes_L V, \quad u_V(v) = 1 \otimes v.$$

It readily follows that all maps described above are well-defined.

**Proposition 5.7.** *The following assertions hold:*

1. *The triple $(L\text{-}\mathrm{mod}, U_K, c)$ is a $T_H$-equivariant structure.*
2. *There is an equivalence $K\text{-}\mathrm{mod} \simeq (L\text{-}\mathrm{mod})^{U_K}$ of $H\text{-}\mathrm{mod}$-module categories.*
3. *$(L\text{-}\mathrm{mod}, U_K, c)$ is simple if and only if $K$ is $H$-simple.*

*Proof.* 1. The proof that $\nu_V : U_K \circ U_K \to U_K$ is a module natural transformation is straightforward. One has to observe that the module structure on the functor $U_K \circ U_K$, given in Lemma 4.1, is

$$d_{X,V} : K \otimes_L K \otimes_L (X \otimes_L V) \to (H \otimes_{L^e} X) \otimes_L (K \otimes_L K \otimes_L V),$$

$$d_{X,V}(h\otimes g\otimes x\otimes v) = h_{(-1)}g_{(-1)}\otimes x\otimes h_{(0)}\otimes g_{(0)}\otimes v,$$

for all $h, g \in K, x \in X, v \in V$. The proof of part 2 is straightforward.

3. Assume there exists a non-trivial $H$-costable ideal $I \subseteq K$. The canonical projection $K \to K/I$ induces a natural morphism $U_{K/I} \to U_K$ that it is not an isomorphism. Hence $(L\text{-}\operatorname{mod}, U_K, c)$ is not simple. If $K$ is $H$-simple then, by the argument in the proof of [2, Proposition 1.18], the module category $(L\text{-}\operatorname{mod})^{U_K}$ is simple. The result follows from part 2 and Theorem 4.9 (3). ☐

**Theorem 5.8.** *Assume that the algebra $L$ is simple. Any exact indecomposable module category over $H$-mod is equivalent to $K$-mod for some $H$-simple left $H$-comodule algebra $(K, s_K, \lambda)$.*

*Proof.* It follows from Corollary 4.13 that any exact indecomposable module category over $H\text{-}\operatorname{mod} \simeq (L\text{-}\operatorname{mod}\text{-}L)^{T_H}$ is of the form $\mathcal{N}^U$ for some exact indecomposable $L$-mod-$L$-module category $\mathcal{N}$ and a simple $T_H$-equivariant structure $(\mathcal{N}, U, c)$. Since $L$ is simple as an algebra, the only exact indecomposable $L$-mod-$L$-module category is $L$-mod, hence $\mathcal{N} \simeq L\text{-}\operatorname{mod}$. Since $U : L\text{-}\operatorname{mod} \to L\text{-}\operatorname{mod}$ is an exact functor, there exists an $L$-bimodule $K$ such that $U(V) = K\otimes_L V$ for all $V \in L\text{-}\operatorname{mod}$. Let us prove that $K$ is a left $H$-comodule algebra.

Define $s_K : L \to K, s_K(l) = l \cdot k$ for all $l \in L, k \in K$. For any $X \in L\text{-}\operatorname{mod}\text{-}L, V \in L\text{-}\operatorname{mod}$ the module structure on $U$ is given by the map

$$c_{X,V} : K\otimes_L(X\otimes_L V) \to (H\otimes_{L^e} X)\otimes_L K\otimes_L V.$$

Define $\lambda : K \to H\otimes_L K$ by $\lambda(k) = c_{L,L}(k\otimes 1\otimes 1)$ for all $k \in K$. Equations (5.11) follows from (2.3) and (2.4). Since $U$ is a monad, there exists a module transformation $\mu : U^2 \to U$. The algebra structure on $K$ is given by $\mu_L : K\otimes_L K \to K$. Equations (5.13) follows since $\mu$ is a module transformation. It follows from Proposition 5.7 that there is a module equivalence $\mathcal{N}^U \simeq K\text{-}\operatorname{mod}$. ☐

## 6. A 2-categorical interpretation

Let us briefly recall the notion of 2-monad in a 2-category. The reader is refered to [16], [25]. A *2-category* consists of

- a class of objects or 0-cells $Obj(\mathcal{B})$;
- for a pair of 0-cells $A, B$ a category $\mathcal{B}(A, B)$. Objects in $\mathcal{B}(A, B)$ are 1-cells and morphisms are called 2-cells;
- for any 0-cell $A$ there is a 1-cell $I_A \in \mathcal{B}(A, A)$;
- for any 0-cells $A, B, C$ a functor
$$\circ^{A,B,C} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C),$$
such that it is associative and unitary. Sometimes we shall omit the superscript and denote the functor $\circ^{A,B,C}$ simply as $\circ$.

If $\mathcal{B}, \mathcal{B}'$ are 2-categories, a *2-functor* $F : \mathcal{B} \to \mathcal{B}'$ consists of the following data:

- an assignment $F : Obj(\mathcal{B}) \to Obj(\mathcal{B}')$;
- for any 0-cells $A, B$ a functor $F_{A,B} : \mathcal{B}(A,B) \to \mathcal{B}'(F(A), F(B))$ such that for any 0-cells $A, B, C$ and 1-cells $X \in \mathcal{B}(B,C), Y \in \mathcal{B}(A,B)$

$$F_{A,C}(X \circ Y) = F_{B,C}(X) \circ F_{A,B}(Y), \quad F_{A,A}(I_A) = I'_{F(A)}.$$

If $X \in \mathcal{B}(A,B)$ is a 1-cell or a 2-cell we shall sometimes denote $F_{A,B}(X)$ simply by $F(X)$ avoiding subscripts.

If $\mathcal{B}, \mathcal{B}'$ are 2-categories and $F, G : \mathcal{B} \to \mathcal{B}'$ are 2-functors, a *2-natural transformation* $\theta : F \to G$ consists of the following data:

- for any 0-cell $A \in Obj(\mathcal{B})$ a 1-cell $\theta_A \in \mathcal{B}'(F(A), G(A))$;
- for any 0-cells $A, B \in Obj(\mathcal{B})$ and any 1-cell $X \in \mathcal{B}(A,B)$ a natural transformation

$$\theta_X : \theta_B \circ F_{A,B}(X) \to G_{A,B}(X) \circ \theta_A,$$

  such that for any 0-cell $A$ and any 1-cells $X, Y$

$$\theta_{X \circ Y} = (\mathrm{id} \circ \theta_Y)(\theta_X \circ \mathrm{id}), \quad \theta_{I_A} = \mathrm{id}_{\theta_A}.$$

**Definition 6.1.** 1. Let $\mathcal{B}$ be a 2-category. A *2-monad* over $\mathcal{B}$ is a strict monad, in the sense of [3, Definition 5.4.1], inside the 2-category of 2-categories. Explicitly, a 2-monad is a collection $(\mathfrak{T}, \mu, \eta)$ where $\mathfrak{T} : \mathcal{B} \to \mathcal{B}$ is a 2-functor, $\mu : \mathfrak{T}^2 \to \mathfrak{T}$ and $\eta : \mathrm{Id} \to \mathfrak{T}$ are 2-natural transformations satisfying

$$\mu_A \circ \mu_{\mathfrak{T}(A)} = \mu_A \circ \mathfrak{T}_{\mathfrak{T}^2(A), \mathfrak{T}(A)}(\mu_A), \quad \mu_A \circ \mathfrak{T}_{A, \mathfrak{T}(A)}(\eta_A) = I_{\mathfrak{T}(A)} = \mu_A \circ \eta_{\mathfrak{T}(A)},$$

$$\mu_X \mu_{\mathfrak{T}(X)} = \mu_X \mathfrak{T}(\mu_X), \quad \mu_X \eta_{\mathfrak{T}(X)} = \mathrm{id}_{\mathfrak{T}(X)} = \mu_X \mathfrak{T}(\eta_X),$$

for any 0-cells $A, B$, and any 1-cell $X \in \mathcal{B}(A,B)$.

**Example 6.2.** Let $\mathcal{C}$ be a strict monoidal category. Associated to $\mathcal{C}$ there is a 2-category $\underline{\mathcal{C}}$ with a single object 0. Namely, $\underline{\mathcal{C}}(0,0) = \mathcal{C}$ and the composition is the monoidal product in $\mathcal{C}$. A bimonad $T : \mathcal{C} \to \mathcal{C}$, with strict comonoidal structure, gives rise to a 2-monad $\underline{T} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$; $\underline{T}(0) = 0$ and $\underline{T}_{0,0} = T$.

If $\mathcal{C}$ is a tensor category, we shall denote by ${}_{\mathcal{C}}\mathrm{Mod}$, respectively ${}_{\mathcal{C}}\mathrm{Mod}^{lax}$, the 2-categories whose 0-cells are left $\mathcal{C}$-module categories, 1-cells are $\mathcal{C}$-module functors (respectively, lax $\mathcal{C}$-module functors) and 2-cells are $\mathcal{C}$-module natural transformations.

Let $T$ be a Hopf monad on $\mathcal{C}$. Define the 2-functor $\mathfrak{T} : {}_{\mathcal{C}}\mathrm{Mod}^{lax} \to {}_{\mathcal{C}}\mathrm{Mod}^{lax}$ as follows. For any $\mathcal{M} \in {}_{\mathcal{C}}\mathrm{Mod}^{lax}$, set $\mathfrak{T}(\mathcal{M}) = \mathcal{M}(T)$, see Subsection 4.1. For any pair $\mathcal{M}, \mathcal{N} \in {}_{\mathcal{C}}\mathrm{Mod}^{lax}$, set

$$\mathfrak{T}_{\mathcal{M},\mathcal{N}} : \mathrm{Hom^{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom^{lax}}_{\mathcal{C}}(\mathcal{M}(T), \mathcal{N}(T))$$

to be the functor defined by

$$\mathfrak{T}_{\mathcal{M},\mathcal{N}}(G, d) = (G, d_{T(-),-}).$$

**Lemma 6.3.** *The 2-functor $\mathfrak{T} : {}_{\mathcal{C}}\mathrm{Mod}^{lax} \to {}_{\mathcal{C}}\mathrm{Mod}^{lax}$ has a structure of 2-monad.*

*Proof.* We shall define 2-natural transformations $\mu : \mathfrak{T}^2 \to \mathfrak{T}, \eta : \mathrm{Id} \to \mathfrak{T}$ such that $(\mathfrak{T}, \mu, \eta)$ is a 2-monad. Note that, abusing of the notation, we are denoting with the same symbols the 2-monad structure on $\mathfrak{T}$ and the monad structure on $T$.

For any $\mathcal{M} \in {}_{\mathcal{C}}\mathrm{Mod}^{lax}$ define $\eta_{\mathcal{M}} \in \mathrm{Hom}^{\mathrm{lax}}{}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}(T))$ the lax $\mathcal{C}$-module functor as $\eta_{\mathcal{M}} = (\mathrm{Id}_{\mathcal{M}}, \eta\overline{\otimes}\mathrm{id}\,)$. Here the module structure of the identity functor is given by

$$\eta_X\overline{\otimes}\mathrm{id}\,_M : X\overline{\otimes}M \to T(X)\overline{\otimes}M,$$

for any $X \in \mathcal{C}, M \in \mathcal{M}$. It follows from (3.6) that $(\mathrm{Id}_{\mathcal{M}}, \eta\overline{\otimes}\mathrm{id}\,)$ is indeed a module functor. To give a structure of 2-natural transformation on $\eta$, for any $(G, d) \in \mathrm{Hom}^{\mathrm{lax}}{}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, we must define natural transformations

$$\eta_{(G,d)} : \eta_{\mathcal{M}} \circ (G, d) \to \mathfrak{T}_{\mathcal{M},\mathcal{N}}(G, d) \circ \eta_{\mathcal{N}}.$$

Since both functors are equal, we let $\eta_{(G,d)}$ to be the identity natural transformation. Now, let us define the 2-natural transformation $\mu : \mathfrak{T}^2 \to \mathfrak{T}$. For any $\mathcal{M} \in {}_{\mathcal{C}}\mathrm{Mod}^{lax}$, let $\mu_{\mathcal{M}} = (\mathrm{Id}_{\mathcal{M}}, \mu\overline{\otimes}\mathrm{id}\,)$. It follows from (3.5) that this functor is indeed a module functor. For any $\mathcal{M}, \mathcal{N} \in {}_{\mathcal{C}}\mathrm{Mod}^{lax}$, $(G, d) \in \mathrm{Hom}^{\mathrm{lax}}{}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, we must define natural transformations

$$\mu_{(G,d)} : \mu_{\mathcal{M}} \circ \mathfrak{T}^2_{\mathcal{M},\mathcal{N}}(G, d) \to \mathfrak{T}_{\mathcal{M},\mathcal{N}}(G, d) \circ \mu_{\mathcal{N}}.$$

Since both functors are equal, we define $\mu_{(G,d)}$ the identity natural transformation. Conditions of Definition 6.1 are readily verified. □

In the next subsection we shall give an interpretation of the process of equivariantization of module categories in the form of a 2-category equivalence between the 2-category ${}_{\mathcal{C}}\mathrm{Mod}^{lax}$ with an appropriate equivariantization of the 2-category of $T$-equivariant lax $\mathcal{C}$-module categories, that we define next.

**Definition 6.4.** Let $\mathcal{C}$ be a tensor category and $T : \mathcal{C} \to \mathcal{C}$ be a Hopf monad. The 2-category ${}_{\mathcal{C}}\mathrm{EqMod}$ of $T$-equivariant lax $\mathcal{C}$-module categories is defined as follows: 0-cells are $T$-equivariant $\mathcal{C}$-module categories $(\mathcal{M}, U, c)$. If $(\mathcal{M}, U, c)$, $(\widetilde{\mathcal{M}}, \widetilde{U}, \widetilde{c})$ are $T$-equivariant $\mathcal{C}$-module categories, 1-cells are pairs $(G, \theta) : (\mathcal{M}, U, c) \to (\widetilde{\mathcal{M}}, \widetilde{U}, \widetilde{c})$, where $G : \mathcal{M} \to \widetilde{\mathcal{M}}$ is a $\mathcal{C}$-module functor and $\theta : \widetilde{U} \circ G \to \widehat{T}(G) \circ U$ is a natural transformation satisfying condition (4.5), that is,

$$\theta_M\widetilde{\nu}_{G(M)} = G(\nu_M)\theta_{U(M)}\widetilde{U}(\theta_M), \quad \theta_M\widetilde{u}_{G(M)} = G(u_M),$$

for all $M \in \mathcal{M}$. If $(H, \chi)$ is another 1-cell, a 2-cell $\alpha : (G, \theta) \Rightarrow (H, \chi)$ is a $\mathcal{C}$-module natural transformation $\alpha : G \to H$ satisfying condition (4.6), that is,

$$\alpha_{U(M)}\theta_M = \chi_M\widetilde{U}(\alpha_M),$$

for all $M \in \mathcal{M}$.

6.1. **Equivariantization of 2-categories.** Let $\mathcal{B}$ be a 2-category and let $(F, \mu, \eta) : \mathcal{B} \to \mathcal{B}$ be a 2-monad on $\mathcal{B}$. We start by giving a description of the 2-category $\mathcal{B}^F$ of $F$-equivariant objects in $\mathcal{B}$.

The horizontal composition of 1 or 2-cells in the 2-category $\mathcal{B}$ will be denoted by $\circ$, omitting the superscripts, and the vertical composition of 2-cells will be indicated by juxtaposition of morphisms.

**Definition 6.5.** An *equivariant object* (or *equivariant 0-cell*) is a collection $(A, U, \nu, u)$ where

- $A$ is a 0-cell in $\mathcal{B}$;
- $U : A \to F(A)$ is a 1-cell in $\mathcal{B}$;
- $\nu : \mu_A \circ F(U) \circ U \Rightarrow U$, $u : \eta_A \Rightarrow U$ are 2-cells, such that

$$\nu(\mathrm{id}_{\mu_A \circ F(U)} \circ \nu)(\mathrm{id}_{\mu_A} \circ \mu_U \circ \mathrm{id}_{F(U)\circ U}) = \nu(\mathrm{id}_{\mu_A} \circ F(\nu) \circ \mathrm{id}_U), \tag{6.1}$$

$$\nu(\mathrm{id}_{\mu_A} \circ F(u) \circ \mathrm{id}_U) = \mathrm{id}_U = \nu(\mathrm{id}_{\mu_A \circ F(U)} \circ u)(\mathrm{id}_{\mu_A} \circ \eta_U). \tag{6.2}$$

Let $(A, U, \nu, u)$, $(\widetilde{A}, \widetilde{U}, \widetilde{\nu}, \widetilde{u})$ be $F$-equivariant objects. An *equivariant 1-cell* $(\theta, \theta^0) : (A, U, \nu, u) \to (\widetilde{A}, \widetilde{U}, \widetilde{\nu}, \widetilde{u})$ consists of

- a 1-cell $\theta : A \to \widetilde{A}$;
- a 2-cell $\theta^0 : \widetilde{U} \circ \theta \Rightarrow F(\theta) \circ U$,

satisfying the following conditions:

$$\theta^0(\widetilde{u} \circ \mathrm{id}_\theta) = (\mathrm{id}_{F(\theta)} \circ u)\eta_\theta,$$

$$(\mathrm{id}_{F(\theta)} \circ \nu)(\mu_\theta \circ \mathrm{id}_{F(U)\circ U})(\mathrm{id}_{\mu_{\widetilde{A}}} \circ F(\theta^0) \circ \mathrm{id}_U)(\mathrm{id}_{\mu_{\widetilde{A}} \circ F(\widetilde{U})} \circ \theta^0) = \theta^0(\widetilde{\nu} \circ \mathrm{id}_\theta).$$

Let $(A, U, \nu, u)$ and $(\widetilde{A}, \widetilde{U}, \widetilde{\nu}, \widetilde{u})$ be $F$-equivariant objects and let $(\theta, \theta^0)$, $(\chi, \chi^0) : (A, U, \nu, u) \to (\widetilde{A}, \widetilde{U}, \widetilde{\nu}, \widetilde{u})$ be equivariant 1-cells. An *equivariant 2-cell* $(\theta, \theta^0) \Rightarrow (\chi, \chi^0)$ is a 2-cell $\alpha : \theta \Rightarrow \chi$ such that

$$\chi^0(\mathrm{id}_{\widetilde{U}} \circ \alpha) = (F(\alpha) \circ \mathrm{id}_U)\theta^0. \tag{6.3}$$

The proof of the following proposition is left to the reader.

**Proposition 6.6.** *Equivariant 0-cells, equivariant 1-cells and equivariant 2-cells form a 2-category with respect to composition of 1-cells* $(\theta, \theta^0) : (A, U, \nu, u) \to (\widetilde{A}, \widetilde{U}, \widetilde{\nu}, \widetilde{u})$ *and* $(\chi, \chi^0) : (A', U', \nu', u') \to (A, U, \nu, u)$ *defined by*

$$(\theta, \theta^0) \circ (\chi, \chi^0) = (\theta \circ \chi, (\mathrm{id} \circ \chi^0)(\theta^0 \circ \mathrm{id})), \tag{6.4}$$

*and vertical and horizontal compositions of equivariant 2-cells given as the corresponding compositions in the 2-category $\mathcal{B}$.* □

This 2-category will be called the *2-category of $F$-equivariant objects* in $\mathcal{B}$ and will be denoted by $\mathcal{B}^F$.

*Remark* 6.7. An $F$-equivariant object in $\mathcal{B}$ could be explained alternatively as a monad, in the sense of [3, Definition 5.4.1], inside the Kleisli 2-category associated with $F$, and the 2-category $\mathcal{B}^F$ as the 2-category of monads inside the Kleisli 2-category.

The following result is a straightforward application of the definitions.

**Proposition 6.8.** *Let $\mathcal{C}$ be a tensor category and $T:\mathcal{C}\to\mathcal{C}$ be a Hopf monad. Let $\mathfrak{T}: {}_{\mathcal{C}}\mathrm{Mod}^{lax}\to {}_{\mathcal{C}}\mathrm{Mod}^{lax}$ be the 2-monad associated to $T$ described in Lemma 6.3. There exists a 2-equivalence of 2-categories*

$$ {}_{\mathcal{C}}\mathrm{EqMod} \simeq ({}_{\mathcal{C}}\mathrm{Mod}^{lax})^{\mathfrak{T}}. $$

## 7. Module categories and exact sequences

Let $\mathcal{C}$, $\mathcal{D}$ be tensor categories over $\Bbbk$. Recall that a *normal* tensor functor $F:\mathcal{C}\to\mathcal{D}$ is a tensor functor such that for any object $X$ of $\mathcal{C}$, there exists a subobject $X_0\subset X$ such that $F(X_0)$ is the largest trivial subobject of $F(X)$.

If the functor $F$ has a right adjoint $R$, then $F$ is normal if and only if $R(\mathbf{1})$ is a trivial object of $\mathcal{C}$ [7, Proposition 3.5].

Let $\mathcal{C}',\mathcal{C},\mathcal{C}''$ be tensor categories over $\Bbbk$. A sequence of tensor functors

$$ \mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}'' \tag{7.1} $$

is called and *exact sequence of tensor categories* if the following hold:

- The tensor functor $F$ is dominant and normal;
- The tensor functor $f$ is a full embedding;
- The essential image of $f$ is $\mathfrak{Ker}_F$.

See [7]. Here, $\mathfrak{Ker}_F$ is the full tensor subcategory $F^{-1}(\langle\mathbf{1}\rangle)\subseteq\mathcal{C}$ of objects $X$ of $\mathcal{C}$ such that $F(X)$ is a trivial object of $\mathcal{C}''$.

Suppose (7.1) is an exact sequence of tensor categories. Since the functor $F$ is normal, then it induces a fiber functor $\omega_F:\mathcal{C}'\to\mathrm{vect}_{\Bbbk}$ in the form $\omega_F(X)=\mathrm{Hom}_{\mathcal{C}''}(\mathbf{1},Ff(X))$.

The *induced Hopf algebra* $H$ of the exact sequence (7.1) is defined as the coend of the fiber functor $\omega_F:\mathcal{C}'\to\mathrm{vect}_k$, that is, $H=\int^{X\in\mathcal{C}'}\omega_F(X)^{\vee}\otimes\omega_F(X)$. In particular, we have an equivalence of tensor categories $\mathcal{C}'\simeq$ comod-$H$. See [7, Subsection 3.3].

Recall that a $\Bbbk$-linear right exact Hopf monad $T$ on a tensor category $\mathcal{C}''$ is called *normal* if $T(\mathbf{1})$ is a trivial object of $\mathcal{C}''$. By [7, Theorem 5.8] exact sequences (7.1) with finite dimensional induced Hopf algebra $H$ are classified by normal faithful right exact $\Bbbk$-linear Hopf monads $T:\mathcal{C}''\to\mathcal{C}''$, such that the Hopf monad of the restriction of $T$ to the trivial subcategory of $\mathcal{C}''$ is isomorphic to $H$.

### 7.1. The Hopf monad of a Hopf algebra extension.

Consider an exact sequence of finite dimensional Hopf algebras

$$ \Bbbk\longrightarrow K\xrightarrow{i}H\xrightarrow{\pi}\overline{H}\longrightarrow\Bbbk. \tag{7.2} $$

In view of [7, Proposition 3.9] (7.2) induces an exact sequence of finite tensor categories

$$ \text{mod-}\overline{H}\xrightarrow{\pi^*}\text{mod-}H\xrightarrow{i^*}\text{mod-}K. \tag{7.3} $$

In this subsection we shall give an explicit description of the normal Hopf monad $T$ on mod-$K$ corresponding to the exact sequence (7.3) in terms of the cohomological data classifying the Hopf algebra extension (7.2).

As a consequence of the Nichols-Zoeller freeness theorem, the exact sequence (7.2) is cleft. Therefore there exist maps

$$. : \overline{H} \otimes K \to K, \qquad \sigma : \overline{H} \otimes \overline{H} \to K, \tag{7.4}$$

$$\rho : \overline{H} \to \overline{H} \otimes K, \qquad \tau : \overline{H} \to K \otimes K, \tag{7.5}$$

obeying the compatibility conditions in [1, Theorem 2.20], such that $H$ is isomorphic as a Hopf algebra to the bicrossed product $K^\tau\#_\sigma\overline{H}$. Recall that the structure of $K^\tau\#_\sigma\overline{H}$ is determined by the data $(., \sigma, \rho, \tau)$ as follows:

$$\begin{aligned}
&a\#x \,.\, b\#y = a(x_{(1)}.b)\sigma(x_{(2)}, y_{(1)})\#x_{(3)}, y_{(2)},\\
&\Delta(a\#x) = \Delta(a)\tau(x_{(1)})\,(1\#(x_{(2)})_{(0)} \otimes (x_{(2)})_{(1)}\#x_{(3)}),\\
&1_{K^\tau\#_\sigma\overline{H}} = 1\#1, \quad \epsilon(a\#x) = \epsilon(a)\epsilon(x),
\end{aligned}$$

for all $a, b \in K$, $x, y \in \overline{H}$, where we use Sweedler's notation $\rho(x) = x_{(0)} \otimes x_{(1)} \in \overline{H} \otimes K$, for every $x \in \overline{H}$.

In what follows we shall use the identifications $H = K^\tau\#_\sigma\overline{H}$, $K \simeq K\#1 \subseteq H$ and $\pi = \epsilon \otimes \mathrm{id} : H \to \overline{H}$. In this way, the normal tensor functor $F = i^* :$ mod-$H \to$ mod-$K$ corresponds to the restriction functor $\mathrm{Res}\,^H_K$.

It is well known that the induction functor $L = \mathrm{Ind}\,^H_K :$ mod-$K \to$ mod-$H$ is left adjoint of $F$, where for every right $K$-module $W$, $\mathrm{Ind}\,^H_K(W) = W \otimes_K H$. It is clear from the formula defining the multiplication of $K^\tau\#_\sigma\overline{H}$ that $H \simeq K \otimes \overline{H}$ as left $K$-modules, where $K$ acts by left multiplication in the first tensorand on $K \otimes \overline{H}$. Hence we obtain natural isomorphisms

$$r_W : L(W) \simeq W \otimes \overline{H}, \quad r_W(w \otimes a\#x) = (w \leftharpoonup a) \otimes x,$$

for every finite-dimensional right $K$-module $W$, $w \in W$, $a \in K$, $x \in \overline{H}$, where $\leftharpoonup : W \otimes K \to W$ denotes the $K$-module structure on $W$.

**Proposition 7.1.** *The normal Hopf monad $T :$ mod-$K \to$ mod-$K$ associated to the exact sequence* (7.3) *is given by $T(W) = W \otimes \overline{H}$, where the $K$-action is defined as*

$$(w \otimes x) \leftharpoonup a = w(x_{(1)}.a) \otimes x_{(2)},$$

*for all $w \in W$, $x \in \overline{H}$, $a \in K$. For all $W, U \in$ mod-$K$, the multiplication $\mu_W : W \otimes \overline{H} \otimes \overline{H} \to W \otimes \overline{H}$, counit $\eta_W : W \to W \otimes \overline{H}$, and comonoidal structure $\xi_{W,U} : W \otimes U \otimes \overline{H} \to W \otimes \overline{H} \otimes U \otimes \overline{H}$, $\phi : \Bbbk \otimes \overline{H} \to \Bbbk$, of $T$ are* *determined, respectively, as follows:*

$$\mu_W(w\otimes x\otimes y) = w \leftharpoonup \sigma(x_{(1)}, y_{(1)}) \otimes x_{(2)}y_{(2)},$$
$$\eta_W(w) = w\otimes 1,$$
$$\xi_{W,U}(w\otimes u\otimes x) = (w\otimes u) \leftharpoonup \tau(x_{(1)})(1\otimes (x_{(2)})_{(1)}) \otimes (x_{(2)})_{(0)} \otimes x_{(3)},$$
$$\phi = \mathrm{id}\, \otimes \epsilon_{\overline{H}} : \Bbbk \otimes \overline{H} \to \Bbbk,$$

*for all* $w\in W$*,* $u\in U$*,* $x,y\in \overline{H}$*.*

*Proof.* Since $F$ : mod-$H$ $\to$ mod-$K$ is a strict strong monoidal functor, then the monad $T = FL$ of the adjunction $(L, F)$ is a bimonad with the prescribed structure; see [9, Theorem 2.6]. Furthermore, $T$ is a Hopf monad, by [6, Proposition 3.5]. Note that, thus defined, $T$ is normal, faithful and right exact and the induced Hopf algebra of $T$ in the sense of [7, Section 5] coincides with $L(\Bbbk)^* = (\overline{H})^*$. This proves the proposition. □

*Remark* 7.2. Consider the case where the exact sequence (7.2) is *cocentral* or, equivalently, $\overline{H}$ is isomorphic to the group algebra $\Bbbk G$ of a finite group $G$ and the weak coaction $\rho$ is trivial. It is known that the cohomological data $., \sigma, \tau$ give rise to an action of $G$ on the category mod-$K$ by tensor autoequivalences and the equivariantization (mod-$K$)$^G$ is equivalent to mod-$H$ as tensor categories [18, Subsection 3.3]. In this case, the normal Hopf monad $T$ associated to the Hopf algebra extension by Proposition 7.1 coincides with the Hopf monad of the corresponding group action given by [7, Theorem 5.21].

7.2. **The category $\mathcal{C}\rtimes T$ when $T$ is normal.** An exact sequence of tensor categories

$$\mathcal{C}' \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}'' \tag{7.6}$$

is called *perfect* if $F$ is a perfect tensor functor, that is, $F$ admits an exact right adjoint $R : \mathcal{C}'' \to \mathcal{C}$.

Suppose that $\mathcal{C}'$ is a finite tensor category. As shown in [7, Theorem 5.8], (7.6) corresponds to a normal faithful $\Bbbk$-linear right exact Hopf monad $T$ on $\mathcal{C}''$. More precisely, since $\mathcal{C}'$ is finite, then the functor $F$ has a left adjoint $L$ and $T = FL$ is the Hopf monad of this adjunction. The normal Hopf monad $T$ gives rise to an exact sequence comod-$H \longrightarrow (\mathcal{C}'')^T \xrightarrow{F} \mathcal{C}''$, where $H$ is the induced Hopf algebra of $T$, and this exact sequence is equivalent to (7.6).

Let us assume that the exact sequence (7.6) is perfect or, equivalently, that $T$ is an exact endofunctor of $\mathcal{C}''$.

Let $(A, \sigma) \in \mathcal{Z}(\mathcal{C})$ be the induced central algebra of $F$. Since $A = R(\mathbf{1})$, then the normality of $F$ is equivalent to the assumption that $A$ belongs to $\mathfrak{Ker}_F$.

We shall use the identifications $\mathcal{C}'' = \mathcal{C}_A$ and $F = F_A : \mathcal{C} \to \mathcal{C}_A$ is the free $A$-module functor. Since $F_A$ is a tensor functor, then $F_A(A) = A\otimes A$ is an algebra in $\mathcal{C}_A$ with multiplication $m\otimes \mathrm{id}_A$.

An object of ${}_{F(A)}(\mathcal{C}_A)$ is a right $A$-module $Y$ in $\mathcal{C}$ endowed with a morphism $F_A(A) \otimes_A Y \to Y$ in $\mathcal{C}_A$. Note that for all object $Y$ of $\mathcal{C}_A$ we have a canonical isomorphism $F_A(A) \otimes_A Y \simeq A \otimes Y$ in $\mathcal{C}_A$, where the right $A$-module structure on $A \otimes Y$ is given by the right action of $A$ on $Y$. Hence a morphism $F_A(A) \otimes_A Y \to Y$ is uniquely determined by a morphism $A \otimes Y \to Y$ of right $A$-modules in $\mathcal{C}$.

We obtain in this way an equivalence of categories ${}_{F(A)}\mathcal{C}'' \simeq {}_A\mathcal{C}_A$. Since $F$ is normal, $F(A)$ is a trivial object of $\mathcal{C}''$, and this restricts in addition to an equivalence ${}_{F(A)}\langle \mathbf{1} \rangle \simeq {}_A(\mathfrak{Ker}_F)_A$.

Let $\mathcal{A}$ and $\mathcal{B}$ be finite abelian categories over $k$. Recall that their Deligne's tensor product $\mathcal{A} \boxtimes \mathcal{B}$, introduced in [10, Section 5], is a finite abelian category, denoted by $\mathcal{A} \boxtimes \mathcal{B}$, endowed with a bifunctor $\boxtimes : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$, $\boxtimes(X, Y) = X \boxtimes Y$, which is right exact in both variables.

The tensor product $\mathcal{A} \boxtimes \mathcal{B}$ is uniquely determined, up to a unique equivalence, by the following universal property: for any bifunctor $\Theta : \mathcal{A} \times \mathcal{B} \to \mathcal{E}$ such that $\Theta$ is right exact in both variables, there exists a unique right exact functor $\theta : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{E}$ such that $\theta \circ \boxtimes = \Theta$.

The bifunctor $\boxtimes : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$ is exact in both variables and satisfies

$$\operatorname{Hom}_{\mathcal{A} \boxtimes \mathcal{B}}(X \boxtimes Y, X' \boxtimes Y') \simeq \operatorname{Hom}_{\mathcal{A}}(X, X') \otimes \operatorname{Hom}_{\mathcal{B}}(Y, Y').$$

Furthermore, any bilinear bifunctor $\Theta : \mathcal{A} \times \mathcal{B} \to \mathcal{E}$, such that $\Theta$ is exact in each variable defines an exact functor $\theta : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{E}$.

Suppose that $\mathcal{A} \simeq A\text{-}\operatorname{mod}$ and $\mathcal{B} \simeq B\text{-}\operatorname{mod}$ as $k$-linear categories, where $A$ and $B$ are finite dimensional algebras over $k$. Then there is an equivalence of $k$-linear categories $\mathcal{A} \boxtimes \mathcal{B} \simeq (A \otimes B)\text{-}\operatorname{mod}$. See [10, Proposition 5.3].

**Proposition 7.3.** *Let $\mathcal{C}$ be a finite tensor category and let $T$ be a normal faithful $\Bbbk$-linear exact Hopf monad on $\mathcal{C}$ with induced Hopf algebra $H$. Then there is an equivalence of $\Bbbk$-linear categories $(\mathcal{C} \rtimes T)^{\mathrm{rev}} \simeq H\text{-}\operatorname{mod} \boxtimes \mathcal{C}$.*

*Proof.* The functor $\mathcal{F} : \mathcal{C}^T \to \mathcal{C}$ gives rise to an exact sequence of finite tensor categories $\mathcal{C}' \to \mathcal{C}^T \xrightarrow{\mathcal{F}} \mathcal{C}$ such that $\mathcal{C}' \simeq \text{comod-}H$. Since $T$ is exact by assumption, then $\mathcal{F}$ is a perfect tensor functor and there is an equivalence of tensor categories $\mathcal{C} \rtimes T \simeq_A (\mathcal{C}^T)_A$, where $(A, \sigma)$ is the induced central algebra of $T$.

From the previous discussion, we have that ${}_A(\mathcal{C}^T)_A \simeq {}_{\mathcal{F}(A)}\mathcal{C}$. Since $\mathcal{F}(A)$ is a trivial object of $\mathcal{C}$, then it follows from [10, Proposition 5.11] that the tensor product $\otimes : \langle \mathbf{1} \rangle \times \mathcal{C} \to \mathcal{C}$ induces an equivalence of $k$-linear categories ${}_{\mathcal{F}(A)}\langle \mathbf{1} \rangle \boxtimes \mathcal{C} \to {}_{\mathcal{F}(A)}\mathcal{C}$.

To finish the proof we observe that there is an equivalence of tensor categories ${}_A\mathcal{C}'_A \simeq H\text{-}\operatorname{mod}$. Indeed, the normality of $\mathcal{F}$ implies that $\mathcal{F}$ induces a fiber functor $\mathcal{F} : \mathcal{C}' \to \langle \mathbf{1} \rangle$, whose coend is isomorphic to $H$ and whose induced central algebra is isomorphic to $A$. Hence $\operatorname{vect}_{\Bbbk} \simeq \mathcal{C}'_A$. By [21, Theorem 5] we get equivalences of tensor categories ${}_A\mathcal{C}'_A \simeq \operatorname{End}_{\mathcal{C}'}(\mathcal{M}) \simeq H\text{-}\operatorname{mod}$. This finishes the proof of the proposition. □

**Example 7.4.** (Hopf algebra exact sequences.) Consider an exact sequence of Hopf algebras $\Bbbk \to K \to H \to \overline{H} \to \Bbbk$ and assume that $K$ is finite dimensional. Then $H$ is free as a left (or right) module over $K$ and in particular the sequence is cleft [24, Theorem 2.1 (2)]. By [7, Proposition 3.9] we have an exact sequence of tensor categories

$$\text{comod-}K \to \text{comod-}H \to \text{comod-}\overline{H}. \tag{7.7}$$

Moreover, since comod-$K$ is a finite tensor category, then the exact sequence (7.7) is determined by a normal faithful Hopf monad $T$ on comod-$\overline{H}$.

Observe that $K$ has a natural algebra structure in the category comod-$H$ and, by cleftness, there is an equivalence of comod-$H$-module categories comod-$\overline{H} \simeq (\text{comod-}H)_K$. Hence we obtain an equivalence of tensor categories $(\text{comod-}H) \rtimes T \simeq {}_K(\text{comod-}H)_K$. The last category is equivalent to the category of comodules over the coquasibialgebra $(K^* \bowtie \overline{H}, \varphi)$, where $\varphi$ is an associated Kac 3-cocycle [22, Section 6]. Thus we get an equivalence of tensor categories $(\text{comod-}H) \rtimes T \simeq \text{comod-}(K^* \bowtie \overline{H}, \varphi)$.

**Example 7.5.** (Equivariantization exact sequences.) Let $G$ be a finite group and let $\rho : \underline{G} \to \underline{\text{Aut}}_{\otimes}(\mathcal{C})$ be an action by tensor autoequivalences of $G$ on the finite tensor category $\mathcal{C}$. Let also $\mathcal{C}^G$ denote the corresponding equivariantization.

The $G$-action gives rise to a normal Hopf monad $T = T^\rho$ on $\mathcal{C}$ in such a way that $\mathcal{C}^{T^\rho} \simeq \mathcal{C}^G$ as tensor categories over $\mathcal{C}$ (see Example 3.6). As discussed in [7, Subsection 5.3], we obtain in this way a (central) exact sequence of tensor categories

$$\operatorname{Rep} G \to \mathcal{C}^G \to \mathcal{C}.$$

Suppose that $\mathcal{C}$ is a fusion category. The category $\mathcal{C} \rtimes T^\rho$ and its module categories were studied by Nikshych in [20]. It follows from [20, Proposition 3.2] that $\mathcal{C} \rtimes T^\rho$ is equivalent to the crossed product tensor category $\mathcal{C} \rtimes G$ constructed by Tambara in [27].

*Remark* 7.6. Recall that a fusion category is called *pointed* if all its simple objects are invertible. On the other side, a fusion category $\mathcal{C}$ is called *group-theoretical* if it is Morita equivalent to a pointed fusion category, that is, if there exists an exact (hence semisimple) indecomposable $\mathcal{C}$-module category $\mathcal{M}$ such that $\operatorname{End}_{\mathcal{C}}(\mathcal{M})$ is a pointed fusion category. See [12, Subsection 8.8].

The fact that $\mathcal{C}^G$ is Morita equivalent to the crossed product $\mathcal{C} \rtimes G$ implies immediately that if $\mathcal{C}$ is a pointed fusion category, then any equivariantization $\mathcal{C}^G$ is group-theoretical, because in this case $\mathcal{C} \rtimes G$ is itself a pointed fusion category.

We observe, however, that this feature does not extend to more general (even normal) Hopf monads. Take for instance $H = H_p$ to be the non group-theoretical semisimple Hopf algebra of dimension $4p^2$ in [20, Section

5], where $p$ is an odd prime number. It follows from [20, Proposition 5.2] that there is an exact sequence of Hopf algebras

$$\Bbbk \to \Bbbk\mathbb{Z}_2 \to H \to \overline{H} \to \Bbbk,$$

where $\overline{H}$ is a certain semisimple Hopf algebra introduced by Masuoka. Hence we get an exact sequence of fusion categories

$$\overline{H}\text{-}\operatorname{mod} \to H\text{-}\operatorname{mod} \to \Bbbk^{\mathbb{Z}_2}\text{-}\operatorname{mod}.$$

Therefore the non group-theoretical fusion category $H$-mod is equivalent to the fusion category $(\Bbbk^{\mathbb{Z}_2}\text{-}\operatorname{mod})^T$, where $T$ is the normal Hopf monad on the pointed fusion category $\Bbbk^{\mathbb{Z}_2}$-mod given by Proposition 7.1.

Nevertheless we have the following:

**Proposition 7.7.** *Let $\mathcal{C}$ be a pointed fusion category and $T$ be a normal faithful $\Bbbk$-linear Hopf monad on $\mathcal{C}$. Suppose that the induced Hopf algebra of $T$ is commutative. Then $\mathcal{C}^T$ is a group-theoretical fusion category.*

*Proof.* Since $\mathcal{C}^T$ is an extension of $\mathcal{C}$ by $H$-mod, then $\mathcal{C}^T$ is an integral fusion category of Frobenius-Perron dimension equal to $\dim H\,|G|$, where $G$ is the group of invertible objects of $\mathcal{C}$. See [7, Propositions 4.9 and 4.10]. By [12, Corollary 8.14 and Theorem 8.35], $\mathcal{C} \rtimes T$ is also an integral fusion category of Frobenius-Perron dimension $\dim H\,|G|$.

Let $H$ be the induced Hopf algebra of $T$. By Proposition 7.3, we have an equivalence of $\Bbbk$-linear categories $(\mathcal{C} \rtimes T)^{\mathrm{rev}} \simeq H\text{-}\operatorname{mod} \boxtimes \mathcal{C}$. Since $H$ is commutative, then the category $H$-mod is also pointed. Let $g_1, \dots, g_n$ be the pairwise non-isomorphic invertible objects of $H$-mod, where $n = \dim H$, and let $h_1, \dot{;} h_{|G|}$ be the pairwise non-isomorphic invertible objects of $\mathcal{C}$. Then the simple objects of $H\text{-}\operatorname{mod} \boxtimes \mathcal{C}$ are, up to isomorphism, $g_i \boxtimes h_j$, $1 \leq i \leq n$, $1 \leq j \leq |G|$.

In particular the category $H\text{-}\operatorname{mod} \boxtimes \mathcal{C}$ has $n|G|$ simple objects. Then so does the category $\mathcal{C} \rtimes T$. Since $\mathcal{C} \rtimes T$ is integral, then for every simple object $X \in \mathcal{C} \rtimes T$, we must have $\operatorname{FPdim} X = 1$, that is, $X$ is an invertible object. Hence the category $\mathcal{C} \rtimes T$ is pointed and therefore $\mathcal{C}^T$ is group-theoretical, as claimed. □

Proposition 7.7 allows us to recover the fact that any semisimple Hopf algebra $H$ such that $H$ fits into an exact sequence $\Bbbk \to \Bbbk^\Gamma \to H \to \Bbbk F \to \Bbbk$, where $\Gamma$ and $F$ are finite groups, is group-theoretical. In fact, we have in this case $H\text{-}\operatorname{mod} \simeq (\Bbbk^\Gamma\text{-}\operatorname{mod})^T$, where $T$ is the associated normal Hopf monad. From Proposition 7.1 we have that the induced Hopf algebra of $T$ is the commutative Hopf algebra $\Bbbk^F$.

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