Round null surfaces in Kerr space-time

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GR22, Valencia, Spain, July 8 2019

The Kerr metric in [Boyer and Lindquist, 1967] coordinates is

$$\begin{aligned} ds^2 = & \left(1-\frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2(\theta)}{\Sigma}dt\,d\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \\ & -\left(r^2+a^2+\frac{2a^2mr}{\Sigma}\sin^2(\theta)\right)\sin^2(\theta)d\phi^2; \\ & \Sigma = r^2+a^2\cos(\theta)^2, \qquad \Delta = r^2+a^2-2mr. \end{aligned} \tag{1}$$

From [Carter, 1968], **the most general null geodesic congruence**

$$\begin{aligned} \ell^a = \begin{pmatrix} \dot{t} \\ \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} \Longrightarrow \ell_a = g_{ab}\ell^b = & \; E\,dt_a - \frac{\pm\sqrt{[(r^2+a^2)E - aL_z]^2 - K\Delta}}{\Delta}dr_a \\ & - \left(\pm\sqrt{K - \left[a\,E\,\sin(\theta) - \frac{L_z}{\sin(\theta)}\right]^2}\right)d\theta_a - L_z d\phi_a. \end{aligned} \tag{2}$$

where E, L_z and K (Carter Constant), are conserved quantities along each geodesic. In what follows, we will take ($E = 1$).

Let us consider a surface S, defined by (t =constant and r =constant). We are interested in the limit case, where this surface tends to a sphere at future null infinity $S \longrightarrow S_\infty$; and we call $\left(\theta^*, \tilde{\phi}\right)$ to the values of (θ, ϕ) at S_∞.
Then, the vectors $\left(\frac{\partial}{\partial\theta^*}\right)^b$ and $\left(\frac{\partial}{\partial\tilde{\phi}}\right)^b$ are tangent to S_∞.
In our approach, we elect the congruence ℓ_a which is orthogonal to S_∞, it means:

$$\lim_{\lambda\longrightarrow\infty} g_{ab}\ell^a\left(\frac{\partial}{\partial\theta}\right)^b = -\left(\pm\sqrt{K - \left[a\sin(\theta^*) - \frac{L_z}{\sin(\theta^*)}\right]^2}\right) = 0,$$
$$\lim_{\lambda\longrightarrow\infty} g_{ab}\ell^a\left(\frac{\partial}{\partial\phi}\right)^b = L_z = 0. \tag{3}$$

This imposes a condition over the conserved quantities L_z and K

$$L_z = 0,$$
$$K = a^2\sin(\theta^*)^2 \tag{4}$$

We try to define a null function u, such that in each point of the surface ($u =$ constant), the null congruence ℓ_a is orthogonal to the surface

$$(du)_a = \ell_a. \tag{5}$$

To satisfy eq (5), **the exterior derivative of the one form must vanish**. But, from the asymptotic conditions $L_z = 0$ and $K = a^2 \sin(\theta^*)^2$; one has to consider $K = K(r, \theta)$. Then, the exterior derivative is

$$d(\ell_a) = \pm \left[\frac{1}{2\sqrt{(r^2 + a^2)^2 - K\Delta}} \frac{\partial K}{\partial \theta} d\theta \wedge dr \pm |_h \frac{1}{2\sqrt{K - \left(a \sin(\theta)\right)^2}} \frac{\partial K}{\partial r} d\theta \wedge dr \right] = 0;$$

To summarize, the differential equation for $K(r, \theta)$ is

$$\sqrt{(r^2 + a^2)^2 - K(r,\theta)\Delta}\, \frac{\partial K(r,\theta)}{\partial r} \pm |_h \sqrt{K(r,\theta) - a^2 \sin^2(\theta)}\, \frac{\partial K(r,\theta)}{\partial \theta} = 0. \tag{6}$$

with a boundary condition $K(r = \infty, \theta^*) = a^2 \sin(\theta^*)^2$.

Carter Constant: K and $K(r,\theta)$, no contradiction

From [Carter, 1968], we have that K is constant along each geodesic. But we can easily probe there is no contradiction on relaxing $K = K(r,\theta)$, by simple computing the derivative along each geodesic

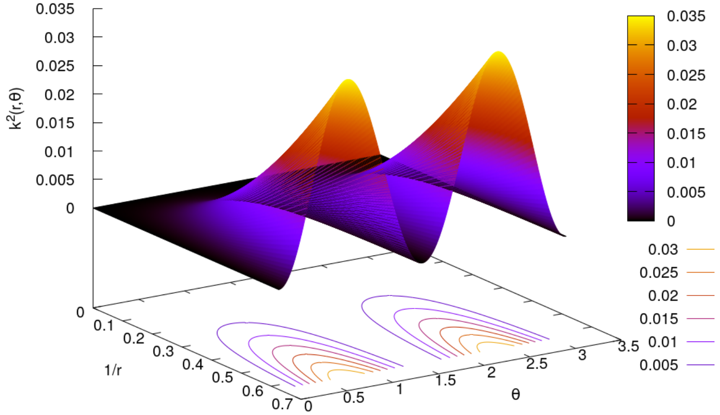
$$\begin{aligned}\frac{dK(r,\theta)}{d\lambda} &= \frac{\partial K}{\partial r}\frac{dr}{d\lambda} + \frac{\partial K}{\partial \theta}\frac{d\theta}{d\lambda} \\ &= \frac{\partial K}{\partial r}\dot{r} + \frac{\partial K}{\partial \theta}\dot{\theta} \\ &= \frac{\partial K}{\partial r}\ell^r + \frac{\partial K}{\partial \theta}\ell^\theta \\ &= \pm\frac{1}{\Sigma}\left[\frac{\partial K}{\partial r}\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta} \pm_h \frac{\partial K}{\partial \theta}\sqrt{K(r,\theta) - a^2\sin^2(\theta)}\right] = 0,\end{aligned} \tag{7}$$

Which is exactly the previous differential equation for $K(r,\theta)$.

It is convenient to define

$$K(r,\theta) = a^2 \sin^2(\theta) + k^2(r,\theta), \tag{8}$$

to see more clearly the asymptotic and interior behavior



Pair of Null Coordinates: u (out-going) and v (in-going)

Once we have the solution $K(r,\theta)$, we actually have a pair of null coordinates, which in [Boyer and Lindquist, 1967] coordinates are

$$d\mathrm{u} = dt - dr_s\left(a, m, r, \theta\right), \tag{9}$$

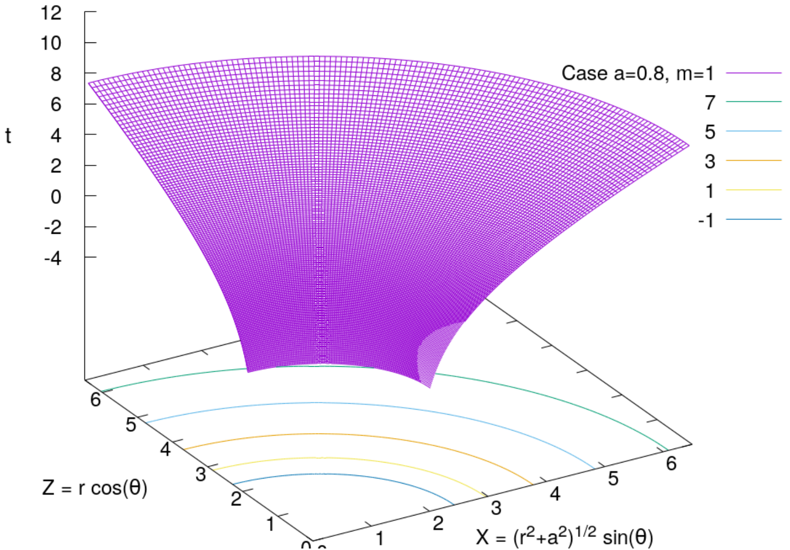
$$d\mathrm{v} = dt + dr_s\left(a, m, r, \theta\right), \tag{10}$$

with

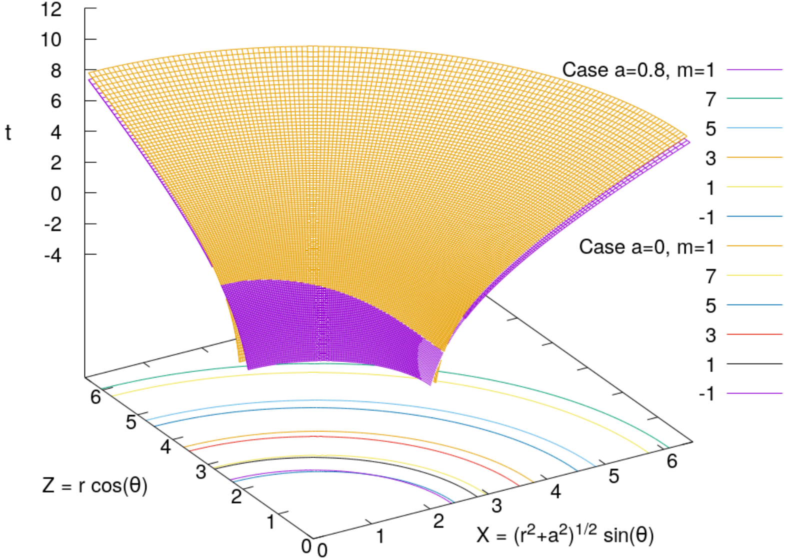
$$dr_s = \frac{\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta}}{\Delta} dr \pm_h \sqrt{K(r,\theta) - a^2\sin^2(\theta)}\, d\theta. \tag{11}$$

To obtain the expressions of u and v, we have to integrate equations (9) and (10). The details of how to make this path-independent integration, together with the final expressions of the null coordinates, will be presented in an up-coming article.

Kerr out-going null coordinate: $(u = 0)$



Kerr and Schwarzschild outgoing null coordinate: $(u = 0)$



ASSOCIATED 2-DIMENSIONAL SPACE-LIKE SURFACE

The intersection of both null coordinates, gives a foliation of 2-D space-like surfaces $(du = 0, dv = 0)$, defined by

$$dr_s = \frac{(d\mathrm{v} - d\mathrm{u})}{2} = 0, \tag{12}$$

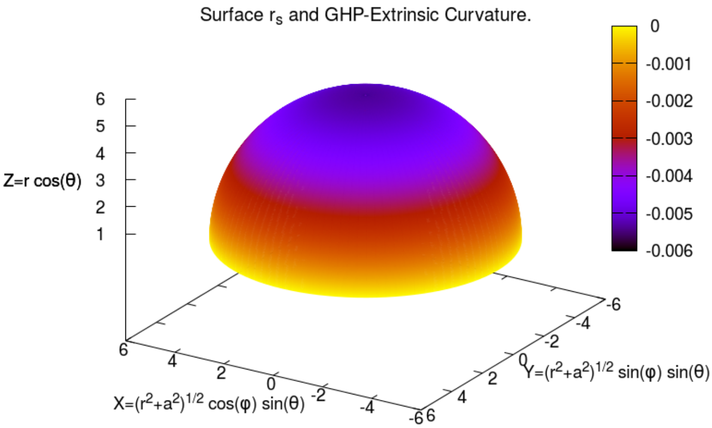
where r_s can be interpreted as the tortoise coordinate (r^*) of Kerr. These surfaces, can be described in a pure geometrically way, in terms of their Gaussian and Extrinsic curvature. From [Geroch et al., 1973], we can compute them by

$$\begin{aligned} K_{Gaussian} &= \left(\bar{Q}_{GHP} + Q_{GHP}\right), \\ K_{Extrinsic} &= i\left(\bar{Q}_{GHP} - Q_{GHP}\right), \end{aligned} \tag{13}$$

$$Q_{GHP} = \sigma\sigma' - \rho\rho' - \Psi_2 + \cancelto{0}{\Lambda} + \cancelto{0}{\Phi_{11}}, \tag{14}$$

where $\sigma, \sigma', \rho, \rho'$ are the spin coefficients in the GHP formalism, and the last two terms become zero in the Kerr case.

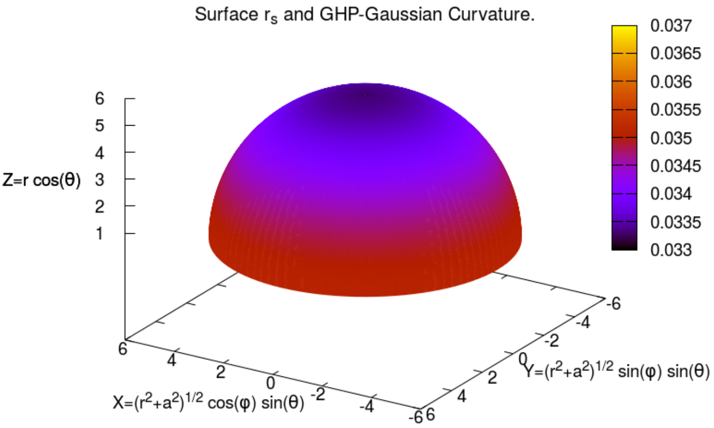
2D surface r_s, $(a = 0.8, m = 1)$: Extrinsic Curvature



In the limit case $(a = 0)$, the surfaces $(r_s = constant) \equiv (r = constant)$:

$$K_{Extrinsic}(a = 0, m, r, \theta) = 0. \qquad \text{(to compare)} \tag{15}$$

2D surface r_s, $(a = 0.8, m = 1)$: Gaussian Curvature



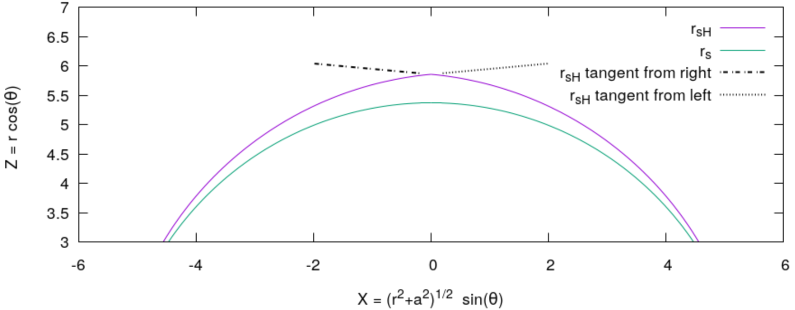
In the limit case $(a = 0)$, the surfaces $(r_s = constant) \equiv (r = constant)$:

$$K_{Gaussian}(a = 0, m, r, \theta) = \frac{1}{r^2}. \qquad \text{(to compare)} \qquad (16)$$

Comparation with other related work: r_s vs r_{sH}

In the PRL article [Hayward, 2004], there is another definition of a Kerr null coordinate system, which differs from ours

$$\left(K = a^2 \quad \text{Hayward definition}\right) \neq \left(K = K(r,\theta) \quad \text{Ours definition}\right)$$



The null hypersurfaces they construct do not include the null geodesics along the axis of symmetry. This is due to the fact that their construction does not give a 2D smooth hypersurface at the poles. One can see that for r_{sH} there is a discontinuity in the derivatives at $(\theta = 0)$, while for r_s it is clearly smooth.

Final Remarks

- We have presented a calculation of Kerr null coordinates which are related to the center of mass frame at future null infinity, sorrounding the black hole all the way up to the horizon.
- This work improves several attempts found in the literature. A remarkable one is [Hayward, 2004].
- Our work is related with the article of [Pretorius and Israel, 1998], which use a very different setting.
- These null coordinates gives a new insight, and we hope it is useful in the study of Kerr solution and the Kerr stability open problem. We plan to use them, in further works of Kerr perturbations.

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