New gravitational lens equations for black holes with angular momentum

Talk by: Ezequiel F. Boero
Collaborator: Osvaldo M. Moreschi

Universidad Nacional de Córdoba,
Inst. de Astronomía Teórica y Experimental, (IATE), CONICET,
Inst. de Física Enrique Gaviola (IFEG), CONICET,
(5000) Córdoba, Argentina

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Outline

Introduction and motivations

- We have been working with general weak lensing formulas [Gallo Moreschi 2011, Boero 2017, Boero Moreschi 2018] in which *optical scalars* are computed by taking into account the whole information of the lens curvature and therefore of its energy-momentum tensor.
- Lenses with angular momentum require a more subtle treatment in comparison with most common static and spherically symmetric ones.
- Expressions for the optical scalars and shear maps are rarely found in the literature[Renzini et al. 2017]; instead a vast amount of works mainly focus on the issue of obtaining an expression for the bending angle and the corresponding thin lens equations [Aazami et al. 2011a, Aazami et al. 2011b]. Also interesting perturbative approaches are found such as [Bozza et al. 2006].
- Interest in the ray tracing techniques on black-holes with angular momentum in numerical studies [Beckwith Done 2005, James et al. 2015, Chen et al. 2015, Chan et al. 2018]
- They has interest by itself at the light of the EHT collaboration announcements of the SMBH in $M87$.

Gravitational lensing I

- The *geodesic deviation equation* for ς^d:

$$\ell^a \nabla_a \left(\ell^b \nabla_b \varsigma^d \right) = R_{abc}{}^{d} \ell^a \varsigma^b \ell^c. \tag{1}$$

- A null tetrad $(\ell^a, m^a, \bar{m}^a, n^a)$ adapted to the central geodesic of a thin bundle leaving the source and reaching the observer
- The geodesic deviation vector $\longrightarrow$ $\varsigma^a = \varsigma \bar{m}^a + \bar{\varsigma} m^a$.
- In components

$$\frac{d^2}{d\lambda^2} \begin{pmatrix} \varsigma \\ \bar{\varsigma} \end{pmatrix} = - \begin{pmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{pmatrix} \begin{pmatrix} \varsigma \\ \bar{\varsigma} \end{pmatrix} ; \tag{2}$$

where the curvature scalars[Geroch 1973] are:

$$\Phi_{00} = -\frac{1}{2} R_{ab}\, \ell^a\, \ell^b \,, \qquad \Psi_0 = C_{abcd}\, \ell^a\, m^b\, \ell^c\, m^d \,; \tag{3}$$

For black-holes $\Phi_{00} = 0$ but $\Psi_0 \neq 0$.

Exact weak lensing and standard weak lensing

- The notion of the optical scalars *optical scalars* $(\kappa, \gamma_1, \gamma_2, \omega)$ comes from the comparison between the non-lensing situation (i.e.: no gravity, and so flat geometry) with respect to the lensing one, due to the curved nature of the spacetime.

Exact solutions to the geodesic deviation equations provide us with a notion exact weak lensing optical scalars, to which approximations can be applied.

Standard weak lensing is normally understood in terms of linear contribution caused by the curvature of the lens.
In common situations the lens is also 'thin' and simple expressions arise [Gallo Moreschi 2011, Boero Moreschi 2018]:

$$\gamma_1 + i\gamma_2 = \frac{1+z_v}{1+z_l}\frac{d_{ls}d_l}{d_s}\int_0^{\lambda_s}\Psi_0 d\lambda. \quad (4)$$

Kerr line element

$$ds^2 = (1-\Phi)\, dt^2 + 2\Phi a \sin^2(\theta)\, dt d\phi - \Sigma\Delta^{-1}\, dr^2 - \Sigma\, d\theta^2 \\ - \left(r^2 + a^2 + \Phi a^2 \sin^2(\theta)\right) \sin^2(\theta)\, d\phi^2; \tag{5}$$

$$\Sigma = r^2 + a^2 \cos(\theta)^2, \qquad \Delta = r^2 - 2rM + a^2, \qquad \Phi = 2Mr\Sigma^{-1}. \tag{6}$$

- It is type-D, so in a *double principal null tetrad* $\left(\tilde{\ell}, \tilde{m}, \bar{\tilde{m}}, \tilde{n}^a\right)$ the curvature is just

$$\widetilde{\Psi}_2 = -\frac{M}{(r - ia\cos(\theta))^3}. \tag{7}$$

Therefore, Ψ_0 must to be proportional to $\widetilde{\Psi}_2$.

- *Geodesic equation* allows enough first integrals $\longrightarrow E, L_z, K$
  So, given a frame (T^a, X^a, Y^a, Z^a) these constants are related to the position and to the angular coordinates (α_x, δ_z) on the sky of the observer.
- The construction of the most natural choice of frame in which Y^a points to the 'center' is not a trivial task to accomplish.
  We propose one based on the *center of mass null congruence* [Argañaraz 2019]

Relating $\widetilde{\Psi}_2$ with Ψ_0

- One is interested in Ψ_0 along each null geodesic bundle.
  The main difficulty being to build the tetrad adapted to the photon paths $(\ell^a, m^a, \bar{m}^a, n^a)$ reaching the observer: ℓ^a is known but m^a seems too hard to find.
- Looking at Lorentz transformations

$$(\tilde{\ell}^a, \tilde{m}^a, \bar{\tilde{m}}^a, \tilde{n}^a) \longrightarrow SO(3,1)^+ \longrightarrow (\ell^a, m^a, \bar{m}^a, n^a); \tag{8}$$

one gets a relation between the tetrad completely described by two real (s, Z) and two complex (Λ, Γ).

$$\ell^a = Z\left(\tilde{\ell}^a + \Lambda\bar{\tilde{m}}^a + \bar{\Lambda}\tilde{m}^a + \Lambda\bar{\Lambda}\tilde{n}^a\right), \tag{9}$$

$$m^a = Z\Gamma\left(\tilde{\ell}^a + \Lambda\bar{\tilde{m}}^a + \bar{\Lambda}\tilde{m}^a + \Lambda\bar{\Lambda}\tilde{n}^a\right) + e^{is}\left(\tilde{m}^a + \Lambda\tilde{n}^a\right), \tag{10}$$

- Curvature transformation reveals that exact knowledge of Ψ_0 just requires to compute the product $Z\Lambda e^{is}$:

$$\boxed{\Psi_0 = 6\left(Z\Lambda e^{is}\right)^2 \widetilde{\Psi}_2.} \tag{11}$$

Conserved quantities for null geodesics in type-D spacetimes

Theorem ([Walker Penrose 1970, Chandrasekhar 1985])

If ℓ^a is an affinely parametrized null geodesic vector and m^a an orthogonal vector to ℓ^a which is parallelly propagated along it; then, in a type-D spacetime the following quantity is conserved along the geodesic:

$$\mathbb{K} = 2\left[(\ell^a \tilde{\ell}_a)(m^a \tilde{n}_a) - (\ell^a \tilde{m}_a)(m^a \bar{\tilde{m}}_a)\right] \widetilde{\Psi}_2^{-1/3}. \tag{12}$$

Corolary

If a restricted Lorentz transformation link tetrad $(\tilde{\ell}^a, \tilde{m}^a, \bar{\tilde{m}}^a, \tilde{n}^a)$ with $(\ell^a, m^a, \bar{m}^a, n^a)$; the conserved quantity along null geodesics $\mathbb{K}$ of the theorem is

$$\mathbb{K} = -2\left(Z\Lambda e^{is}\right) \widetilde{\Psi}_2^{-1/3}. \tag{13}$$

Computation of the lens curvature scalar Ψ_0 III

Very simple expression for Ψ_0

$$\boxed{\Psi_0 = -\frac{3M^{5/3}\mathbb{K}^2}{2\big(r - ia\cos(\theta)\big)^5}.} \tag{14}$$

Remarkably simple formula along any null geodesic in Kerr spacetime; and potentially very useful in the combined numerical integration of the *geodesic* and *geodesic deviation equations*.
Allowing for very efficient computation!

- The constant $\mathbb{K}$ has spin-weight 1 and it's related to the constants of motion (E, L_z, K). In particular, it satisfies [Chandrasekhar 1985]

$$|\mathbb{K}|^2 = 2M^{-2/3}K. \tag{15}$$

Efficient lensing for Kerr spacetime I

Previous expressions vs New expressions

Previous for $\Re(\Psi_0)$

$$
\begin{aligned}
\Re(\Psi_{0*}) = {} & \frac{1}{2(w-1)\big(p^{\hat r}-p^{\hat\imath}\big)^2}\Big\{ Q_1\Big(-3w\big(p_{\hat\theta}^2-p_{\hat\phi}^2\big)^2 \\
& -3p_{\hat r}^4 w + 2p_{\hat r}^3\big[3p^{\hat\imath}\, w + p_{\hat\phi}\, S(w-1)\big] \\
& +6p_{\hat r}^2\Big[p_{\hat\theta}^2 - \big(p^{\hat\imath}\big)^2 w - p^{\hat\imath}\, p_{\hat\phi}\, S(w-1) - p_{\hat\phi}^2(w+1)\Big] \\
& +2p^{\hat r}\Big[p_{\hat\phi}\, S(w-1)\big(p_{\hat\phi}^2-3p_{\hat\theta}^2\big) \\
& +3\big(p^{\hat\imath}\big)^3 w + 3\big(p^{\hat\imath}\big)^2 p_{\hat\phi}\, S(w-1) - 3p^{\hat\imath}\,(w+2)\big(p_{\hat\theta}^2-p_{\hat\phi}^2\big)\Big] \\
& -3\big(p^{\hat\imath}\big)^4 w - 2\big(p^{\hat\imath}\big)^3 p_{\hat\phi}\, S(w-1) \\
& +6\big(p^{\hat\imath}\big)^2\big[p_{\hat\theta}^2(w+1)-p_{\hat\phi}^2\big] - 2p^{\hat\imath}\, p_{\hat\phi}\, S(w-1)\big(p_{\hat\phi}^2-3p_{\hat\theta}^2\big)\Big) \\
& +2\, Q_2\, p_{\hat\theta}\Big(3p_{\hat\phi}\, w\big(p_{\hat\phi}^2-p_{\hat\theta}^2\big) \\
& +p_{\hat r}^3 S(1-w) + 3p_{\hat r}^2\big[p^{\hat\imath}\, S(w-1) + p_{\hat\phi}\,(w+2)\big] \\
& -p_{\hat r}\Big[-S(w-1)\big(p_{\hat\theta}^2-3p_{\hat\phi}^2\big) + 3\big(p^{\hat\imath}\big)^2 S(w-1) \\
& +6p^{\hat\imath}\, p_{\hat\phi}\,(w+2)\Big] + \big(p^{\hat\imath}\big)^3 S(w-1) + 3\big(p^{\hat\imath}\big)^2 p_{\hat\phi}(w+2) \\
& -p^{\hat\imath}\, S(w-1)\big(p_{\hat\theta}^2-3p_{\hat\phi}^2\big)\Big)\Big\},
\end{aligned}
$$

Previous for $\Im(\Psi_0)$

$$
\begin{aligned}
\Im(\Psi_{0*}) = {} & \frac{1}{(w-1)\big(p^{\hat r}-p^{\hat\imath}\big)^2}\Big\{ -Q_2\Big(6p_{\hat\theta}^2 p_{\hat\phi}^2 w + p_{\hat r}^3\big[3p^{\hat\imath}\, w + p_{\hat\phi}\, S(w-1)\big] \\
& +3p_{\hat r}^2\Big[p_{\hat\theta}^2(w+1) - 2\big(p^{\hat\imath}\big)^2 w - p^{\hat\imath}\, p_{\hat\phi}\, S(w-1) - p_{\hat\phi}^2\Big] \\
& +p_{\hat r}\Big[p_{\hat\phi}\, S(w-1)\big(p_{\hat\phi}^2-3p_{\hat\theta}^2\big) \\
& +3\big(p^{\hat\imath}\big)^3 w + 3\big(p^{\hat\imath}\big)^2 p_{\hat\phi}\, S(w-1) - 3p^{\hat\imath}(w+2)\big(p_{\hat\theta}^2-p_{\hat\phi}^2\big)\Big] \\
& -\big(p^{\hat\imath}\big)^3 p_{\hat\phi}\, S(w-1) \\
& +3\big(p^{\hat\imath}\big)^2\big[p_{\hat\theta}^2-p_{\hat\phi}^2(w+1)\big] - p^{\hat\imath}\, p_{\hat\phi}\, S(w-1)\big(p_{\hat\phi}^2-3p_{\hat\theta}^2\big)\Big) \\
& +Q_1\, p_{\hat\theta}\Big(3p_{\hat\phi}\, w\big(p_{\hat\phi}^2-p_{\hat\theta}^2\big) \\
& +p_{\hat r}^3 S(1-w) + 3p_{\hat r}^2\big[p^{\hat\imath}\, S(w-1) + p_{\hat\phi}\,(w+2)\big] \\
& -p_{\hat r}\Big[-S(w-1)\big(p_{\hat\theta}^2-3p_{\hat\phi}^2\big) + 3\big(p^{\hat\imath}\big)^2 S(w-1) \\
& +6p^{\hat\imath}\, p_{\hat\phi}(w+2)\Big] + \big(p^{\hat\imath}\big)^3 S(w-1) + 3\big(p^{\hat\imath}\big)^2 p_{\hat\phi}(w+2) \\
& -p^{\hat\imath}\, S(w-1)\big(p_{\hat\theta}^2-3p_{\hat\phi}^2\big)\Big)\Big\}.
\end{aligned}
$$

Auxiliary functions

$$
Q_1 = \frac{r\big(r^2-3a^2\cos^2\theta\big)}{\rho^6}, \quad Q_2 = \frac{a\cos\theta\big(3r^2-a^2\cos^2\theta\big)}{\rho^6}, \quad w = \frac{\Delta a^2\sin^2\theta}{\big(r^2+a^2\big)^2}, \quad S = \frac{3a\sqrt{\Delta}\big(r^2+a^2\big)\sin\theta}{\Sigma^2}.
$$

Efficient lensing for Kerr spacetime II

Previous expressions vs New expressions

New expression for Ψ_0

$$\Psi_0(r,\theta) = \frac{3M}{(r - ia\cos(\theta))^5}\left[\delta_z r_o - i\left(\sqrt{K_o} + \alpha_x r_o\right)\right]^2 . \tag{16}$$

Shear in thin lens approximation (No need to solve geodesic)

$$\gamma_1 + i\gamma_2 = 3M\frac{d_l d_{ls}}{d_s}\left[\delta_z r_o - i\left(\sqrt{K_o} + \alpha_x r_o\right)\right]^2 \int_0^{\lambda_s} \frac{d\lambda'}{(r - ia\cos(\theta))^5} . \tag{17}$$

Ellipticity maps

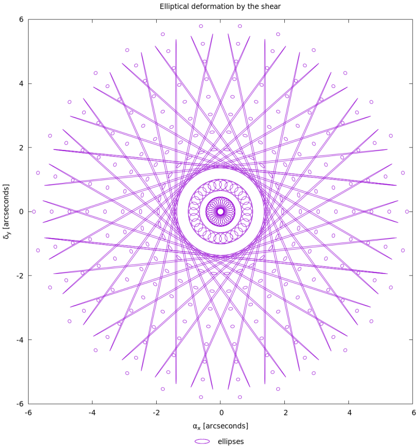

Figure: Elliptical deformation.

Magnification

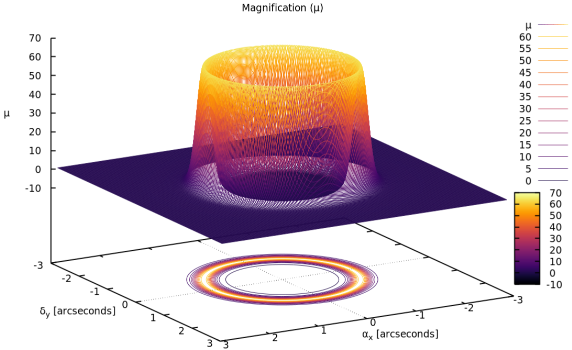

Figure: Predicted magnification.

Summary and final comments

- We showed that recurring to Kerr symmetries one obtains an exact expression for the curvature scalar Ψ_0 present in the *geodesic deviation equation*.
- It is very simple, valid for any geodesic bundle and allow for more efficient calculations in gravitational lensing effects; both weak and exact.
- Our treatment do not recur to the notion of bending angles which is usually the starting point in lensing works and since optical scalars are expressed in terms of curvature they are manifestly gauge invariant.
- We applied to build shear maps in the regime of weak lensing for near extreme Kerr BH with parameters similar to those of M87.

Thanks for your attention!

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Carter's constant I

Theorem

A null geodesic ℓ^a, in any type-D spacetime allows the integral of motion

$$K_0 = 2\left|\tilde{\Psi}_2\right|^{-2/3}\left(\ell^a \tilde{m}_a\right)\left(\ell^a \bar{\tilde{m}}_a\right) = 2\left|\tilde{\Psi}_2\right|^{-2/3}\left(\ell^a \tilde{\ell}_a\right)\left(\ell^a \tilde{n}_a\right). \quad (18)$$

Theorem

The constant K_0 is proportional to the square of the absolute modulus of the constant $\mathbb{K}$ along null geodesic. The proportionality is given by

$$\mathbb{K}\overline{\mathbb{K}} = |\mathbb{K}|^2 = -2\left(m^a \bar{m}_a\right) K_0; \quad (19)$$

where we remember that m^a is an orthogonal vector to null geodesic tangent vector ℓ^a, and parallel propagated the null geodesic.

Relation between $\mathbb{K}$ and K

- The relation between the constant K_0 of the theorem and the Carter's constant K appearing in the geodesic equations is simply a rescaled by the mass of the spacetime appearing trough $\tilde{\Psi}_2$, we mean

$$K = M^{2/3} K_0 \tag{20}$$

- Carter's constant has the advantage to be independent of the mass, and consequently it has a well behaved limit when $M \to 0$.
- Since we take m^a a complex null vector which satisfies $m^a \bar{m}_a = -1$ it follows

$$|\mathbb{K}|^2 = 2M^{-2/3} K. \tag{21}$$

$\mathbb{K}$ as a function of (α_x, δ_z) for distant observers

- The observer lies in a region far from the center $\longrightarrow$ $\frac{a}{r_o} \leq \frac{M}{r_o} \ll 1.$

The contractions needed to compute $\mathbb{K}$ are:

$$\ell^a \tilde{\ell}_a = \frac{K}{2Er_o^2} + \mathscr{O}\left(\frac{KM}{r_o^3}\right), \qquad m_a \tilde{n}^a = \frac{\alpha_x - a\sin(\theta_o) + i\delta_z}{2\sqrt{2}}\left(1 + \mathscr{O}\left(\frac{M}{r_o}\right)\right), \tag{22}$$

$$\ell^a \tilde{m}_a = \frac{-1}{\sqrt{2}r_o}\left[\pm\sqrt{K - \left(\frac{L_z}{\sin(\theta_o)} - \sqrt{K_o}\right)^2} - i\left(\sqrt{K_o} - \frac{L_z}{\sin(\theta_o)}\right)\right]\left(1 + \mathscr{O}\left(\frac{a}{r_o}\right)\right), \tag{23}$$

$$m^a \bar{\tilde{m}}_a = i\left(1 + \mathscr{O}\left(\frac{M}{r_o}\right)\right). \tag{24}$$

and one obtains

$$\boxed{\mathbb{K} = -\frac{i\sqrt{2}}{M^{1/3}}\left[\delta_z r_o - i\left(\sqrt{K_o} + \alpha_x r_o\right)\right]\left(1 + \mathscr{O}\left(\frac{M}{r_o}\right)\right)}; \tag{25}$$

$$\boxed{\Psi_0(r,\theta) = \frac{3M}{(r - ia\cos(\theta))^5}\left[\delta_z r_o - i\left(\sqrt{K_o} + \alpha_x r_o\right)\right]^2\left(1 + \mathscr{O}\left(\frac{M}{r_o}\right)\right)}. \tag{26}$$

Restricted Lorentz transformations

Decomposition of the group $SO^{+}(3,1)$

It is 6-parametric and can be expressed the product of 3 subgroups:

- The 'GHP group' parametrized by two reals (Z, s):

$$\ell^a \to Z\ell^a, \qquad m^a \to m^a e^{is}, \qquad n^a \to Z^{-1} n^a. \tag{27}$$

- The two dimensional $(\Gamma \in \mathbb{C})$ 'null rotation' with ℓ^a as a fixed direction:

$$\ell^a \to \ell^a, \qquad m^a \to m^a + \Gamma \ell^a, \qquad n^a \to n^a + \bar{\Gamma} m^a + \Gamma \bar{m}^a + \Gamma \bar{\Gamma} \ell^a. \tag{28}$$

- The two dimensional $(\Gamma \in \mathbb{C})$ 'null rotation' with n^a as a fixed direction:

$$n^a \to n^a, \qquad m^a \to m^a + \Lambda n^a, \qquad \ell^a \to \ell^a + \bar{\Lambda} m^a + \Lambda \bar{m}^a + \Lambda \bar{\Lambda} n^a. \tag{29}$$

Future repeated principal null tetrad in Kerr

In terms of Boyer-Lindquist coordinates, a common choice for such a null tetrad is the following one [Chandrasekhar 1985]:

$$\tilde{\ell}^a = \frac{r^2 + a^2}{\Delta}\partial_t^a + \partial_r^a + \frac{a}{\Delta}\partial_\phi^a, \tag{30}$$

$$\tilde{n}^a = \frac{r^2 + a^2}{2\Sigma}\partial_t^a - \frac{\Delta}{2\Sigma}\partial_r^a + \frac{a}{2\Sigma}\partial_\phi^a, \tag{31}$$

$$\tilde{m}^a = \frac{ia\sin(\theta)}{\sqrt{2}\mathfrak{r}}\partial_t^a + \frac{1}{\sqrt{2}\mathfrak{r}}\left(\partial_\theta^a + \frac{i}{\sin(\theta)}\partial_\phi^a\right); \tag{32}$$

where the complex function $\mathfrak{r}$ is defined as:

$$\mathfrak{r} = r + ia\cos(\theta), \tag{33}$$

Boyer-Lindquist frame and the Observer frame I

- Null geodesic vector ℓ^a and parallel transport vetor m^a can be written in the tangent space of the observer as:

$$\begin{aligned}\ell^a =& \mathrm{T}^a - \sin\left(\frac{\pi}{2}-\delta_z\right)\cos\left(\frac{\pi}{2}-\alpha_x\right)\mathrm{X}^a \\ &- \sin\left(\frac{\pi}{2}-\delta_z\right)\sin\left(\frac{\pi}{2}-\alpha_x\right)\mathrm{Y}^a - \cos\left(\frac{\pi}{2}-\delta_z\right)\mathrm{Z}^a.\end{aligned} \tag{34}$$

and

$$\begin{aligned}m^a =& \frac{1}{\sqrt{2}}\left[-i\cos\left(\frac{\pi}{2}-\delta_z\right)\cos\left(\frac{\pi}{2}-\alpha_x\right)+\sin\left(\frac{\pi}{2}-\alpha_x\right)\right]\mathrm{X}^a \\ &+ \frac{1}{\sqrt{2}}\left[-i\cos\left(\frac{\pi}{2}-\delta_z\right)\sin\left(\frac{\pi}{2}-\alpha_x\right)-\cos\left(\frac{\pi}{2}-\alpha_x\right)\right]\mathrm{Y}^a + \frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2}-\delta_z\right)\mathrm{Z}^a,\end{aligned} \tag{35}$$

Boyer-Lindquist frame and the Observer frame II

- The frame of the observer $(\mathrm{T}^a, \mathrm{X}^a, \mathrm{Y}^a, \mathrm{Z}^a)$ is chosen as follows:

$$\mathrm{T}_a = \sqrt{1-\mathit{\Phi}_o}dt_a + \frac{\mathit{\Phi}_o a \sin(\theta_o)^2}{\sqrt{1-\mathit{\Phi}_o}} d\phi_a, \tag{36}$$

$$\mathrm{X}_a = \frac{\sqrt{\Delta_o}}{N_\mathrm{y}}\left[\frac{\mathit{\Phi}_o a \sin(\theta_o)}{\sqrt{\Delta_o}\sqrt{1-\mathit{\Phi}_o}}\sqrt{\frac{\Sigma_o}{\Delta_o}}dr_a - \sqrt{\frac{\mathcal{R}_\mathrm{cm}}{\Sigma_o}}\frac{\sin(\theta_o)}{\sqrt{1-\mathit{\Phi}_o}}d\phi_a\right], \tag{37}$$

$$\mathrm{Y}_a = \frac{\sqrt{\Delta_o}}{N_\mathrm{y}}\left[\frac{\sqrt{\mathcal{R}_\mathrm{cm}}}{\Delta_o}dr_a + \frac{\mathit{\Phi}_o a \sin(\theta_o)^2}{1-\mathit{\Phi}_o}d\phi_a\right], \tag{38}$$

$$\mathrm{Z}_a = \sqrt{\Sigma_o}d\theta_a. \tag{39}$$