Black hole equations of motion in the null gauge with back reaction due to radiation

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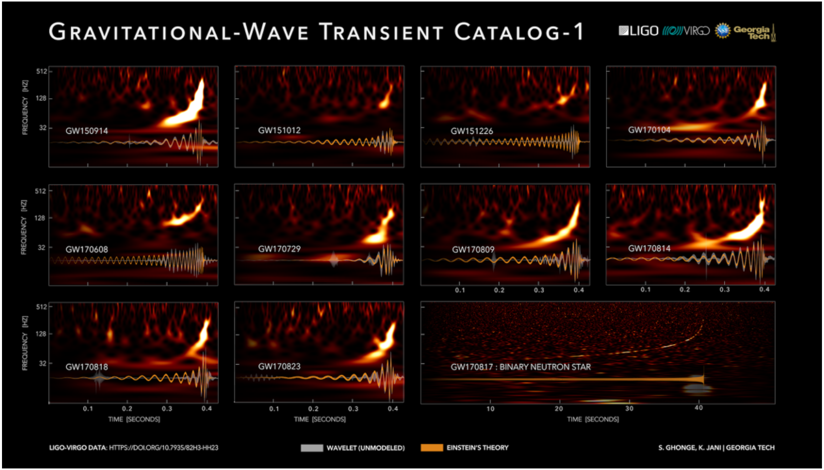
Introduction: I



The black holes in these binary systems have masses in the range from 19 to $85 M_{\odot}$

Introduction: II

With the upcoming new data, it is expected to appear diverse systems with wide range of dynamical parameters and initial conditions.

- There is need for versatile models for black hole binaries which can handle a wide variety of physical conditions.
- The strategy to build equations of motion by requiring to balance the amount of momentum radiated has proved to be useful in the context of charged particles[Gallo and Moreschi(2012)].
- We have presented[Gallo and Moreschi(2019)] the general framework that one must use in order to extend these strategies to relativistic theories of gravity
- We here present the construction of the balanced equations of motion for black holes up to second order in the 'gravitational force' in the null gauge.

The balanced equation of motion approach. I

Basic characteristics of the model for compact objects:

- It is an approximate solution of the Hilbert-Einstein equations.
- The model presents a single spacetime description $(\mathcal{M}, g_{ab})$ of the system.
- The spacetime $(\mathcal{M}, g_{ab})$ is asymptotically flat at future null infinity.
- Each compact object satisfies an equation of motion that takes into account the back reaction due to gravitational radiation.
- For the binary system, the local reference frame and the asymptotic reference frame refers to the appropriate notion of center of mass respectively.
- Each object is described in terms of its relativistic individual dynamical time.

It can be seen that our premises are very different from those of PostNewtonian and self-force approaches. In particular, it is not imposed any weak field or low velocity restrictions. Our construction differs conceptually and in the algebra with the previous approaches.

The balanced equation of motion approach. II

In the asymptotic region (where gravitational radiation can be defined) one can always write the metric as

$$g = \eta_{\mathsf{asy}} + h_{\mathsf{asy}}; \tag{1}$$

where η_{asy} is a flat metric associated to an inertial frame in the asymptotic region and h_{asy} the tensor where all the physical information is encoded. But there are as many flat metrics η_{asy} as there are proper BMS[Sachs(1962), Moreschi(1986)] supertranslations. It is because of this that it is essential to use an unambiguous reference system, given by the center of mass frames.

The notion of rest frames in the asymptotic region, along with that of center of mass(free from supertranslation ambiguities), has been elucidated in previous works: (O.M. Moreschi and S. Dain, *Rest frame system for asymptotically flat space-times*, J.Math.Phys., 39, 12, 6631–6650, 1998), (O.M. Moreschi, *Intrinsic angular momentum and center of mass in general relativity*, Class.Quantum Grav., 21, 5409–5425, 2004), (E. Gallo and O.M. Moreschi, *Intrinsic angular momentum for radiating spacetimes which agrees with the Komar integral in the axisymmetric case*, Phys.Rev., D89, 084009, 2014).

The balanced equation of motion approach. III

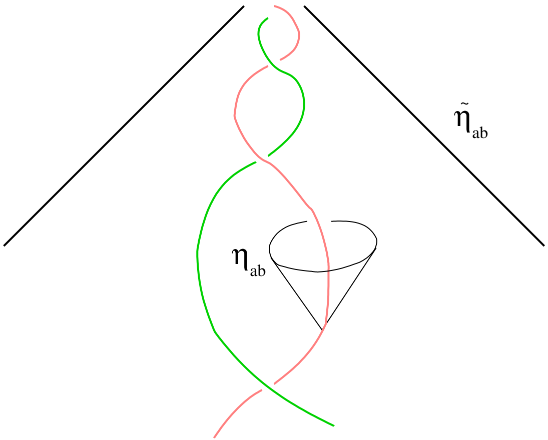
To study the gravitational radiation emitted by the motion of particle A, we will model the asymptotic structure of a sub-metric

$$g_A = \eta + h_A; \tag{2}$$

in order to calculate the reaction due to gravitational radiation.
We adjust g_A to model a compact object, which dynamics is affected by the existence of system B, determined by a sub-metric

$$g_B = \eta + h_B. \tag{3}$$

The balanced equation of motion approach. IV



The application of the general framework in the null gauge to a binary black hole system.

The geometry

If we take the general asymptotic form of an adapted null tetrad of an asymptotically flat spacetime[Moreschi(1987)] and keep only the terms associated with a monopole, then one is left with the line element

$$ds^2 = \left(-2\frac{\dot{V}}{V}r + K_V - 2\frac{M}{r}\right) du^2 + 2\, du\, dr - \frac{r^2}{P^2} d\zeta\, d\bar{\zeta}, \tag{4}$$

where $P = P(u, \zeta, \bar{\zeta}) = V\, P_0$ and

$$K_V = \frac{2}{V}\, \bar{\eth}_V \eth_V\, V - \frac{2}{V^2}\, \eth_V V\, \bar{\eth}_V V + V^2. \qquad \textbf{(??)}$$

We refer to this as the monopole particle line element in the null gauge.
This line element of a monopole can be related to what we have called Robinson-Trautman(RT) geometries[Dain et al.(1996)Dain, Moreschi, and Gleiser]; which are generalizations of Robinson-Trautman spacetimes.
Robinson-Trautman[Robinson and Trautman(1962)] spacetimes have been very useful for estimating the total gravitational radiation in the head-on black hole collision[Moreschi and Dain(1996)][Moreschi(1999)]
[Anninos et al.(1993)Anninos, Hobill, Seidel, Smarr, and Suen].

The equation of motion in the null gauge. II

In reference [Moreschi and Dain(1996)] we have applied these geometries to the description of the total energy radiated in the head-on black hole collision with equal mass; and it was shown that our calculations agree remarkably well with the numerical exact calculations of Anninos et.al.
[Anninos et al.(1993)Anninos, Hobill, Seidel, Smarr, and Suen]. The case of unequal mass black hole collision, was treated numerically in reference
[Anninos and Brandt(1998)]; and our technique based on the use of the RT geometries[Moreschi(1999)] showed again an impressive agreement with the exact calculations.

There is only one Ricci spinor component different from zero, namely

$$\Phi_{22} = \frac{3M\frac{\dot{V}}{V} + \frac{1}{2}\bar{\eth}_V \eth_V K_V}{r^2}. \tag{5}$$

It is important to remark that one would expect $\Phi_{22} = \frac{\Phi^0_{22}}{r^2} + \delta\Phi_{22}(u, r, \zeta, \bar{\zeta})$ where $\delta\Phi_{22}$ decays to zero faster than $\frac{1}{r^2}$ for large r. But $\delta\Phi_{22}(u, r, \zeta, \bar{\zeta}) = 0$!

Next, we will use $(\hat{l}^\alpha) = \big(1, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\big)$.

The equation of motion in the null gauge. III

Global quantities

Let us recall that the Bondi momentum in these geometries can be expressed[Dain et al.(1996)Dain, Moreschi, and Gleiser] as

$$\mathcal{P}^{\alpha} = \frac{1}{4\pi} \int \frac{M}{V^3} \hat{l}^{\alpha} dS^2, \tag{6}$$

It is also very interesting to calculate[Dain et al.(1996)Dain, Moreschi, and Gleiser] the time variation of the total momentum in these geometries. With respect to the instantaneous inertial time $\tilde{u}$ one has

$$\frac{d\mathcal{P}^{\alpha}}{d\tilde{u}} = -\frac{1}{4\pi} \int \left(\frac{\partial \sigma^0}{\partial \tilde{u}} \frac{\partial \bar{\sigma}^0}{\partial \tilde{u}} + \Phi^0_{(I)22} \right) \hat{l}^{\alpha} dS^2; \tag{7}$$

while the time derivative of the total momentum with respect to the intrinsic time is

$$\frac{d\mathcal{P}^{\alpha}}{du} = -\frac{1}{4\pi} \int \left(\frac{\eth^2 V \bar{\eth}^2 V}{V} + \frac{\Phi^0_{22}}{V^3} \right) \hat{l}^{\alpha} dS^2. \tag{8}$$

So we see that actually, we only need to consider the equation

$$\boxed{\int \frac{\Phi^0_{22}}{V^3} \hat{l}^{\alpha} dS^2 = 0}; \tag{9}$$

The equation of motion in the null gauge. IV

which is our main equation for this model that comes both from the interior structure and the asymptotic structure.

It is also convenient to recall the relations between the inertial (Bondi) quantities and the intrinsic ones, namely:

$$\sigma_0' = -\frac{\partial \bar{\sigma}^0}{\partial \tilde{u}} = -\frac{\bar{\eth}^2 V}{V}. \tag{10}$$

There are two natural dynamical times in the interior, for particle A; one is τ, the proper time with respect to the metric g_B, and the other is the proper time τ_0, with respect to the metric η. Let us denote with $\mathbf{v}$ and v be the corresponding tangent vectors to the proper times τ_0 and τ respectively. Then the basic differential operators are $\mathbf{v}^b \partial_b \mathbf{v}^a$ or $v^b \nabla_{(B)b} v^a$; where ∂_b is the covariant derivative associated with the metric η.

Let us note that the two velocity vectors are proportional

$$v^b = \Upsilon \mathbf{v}^b. \tag{11}$$

We have discussed elsewhere[Gallo and Moreschi(2019)] how to construct balanced equations of motion in a general setting applicable to different gauges. Having the flux of momentum, we set the flux force by

$$F(u')^\mu = -\Upsilon \mathcal{F}_V^\mu = -\frac{\Upsilon}{4\pi} \int \frac{\eth^2 V \bar{\eth}^2 V}{V} \hat{l}^\mu dS^2 = \frac{\Upsilon}{4\pi} \int \frac{1}{2V^3} \bar{\eth}_V \eth_V K_V \hat{l}^\mu dS^2; \tag{12}$$

The equation of motion in the null gauge. V

where we are denoting with u' the asymptotic time related to the proper time τ, as explained in [Gallo and Moreschi(2019)].
Then, using the general treatment of the balanced approach[Gallo and Moreschi(2019)], one can write the equations of motion, in first order of the 'gravitational force', as

$$\mathbf{a}^a = \mathbf{f}^a + f_\lambda^a, \tag{13}$$

where

$$\mathbf{a}^a \equiv \mathbf{v}^b \partial_b \mathbf{v}^a, \tag{14}$$

$$\mathbf{f}^b \equiv -\gamma_a{}^b{}_c \, \mathbf{v}^a \, \mathbf{v}^c - \frac{1}{\Upsilon} \frac{d\Upsilon}{d\tau_0} \, \mathbf{v}^b - \frac{w}{\Upsilon} \mathbf{v}^b, \tag{15}$$

and f_λ^μ is defined by

$$M f_\lambda^\mu \equiv \frac{1}{\Upsilon} \mathbf{F}_0^\mu; \tag{16}$$

with

$$\mathbf{F}_0^\mu = -\frac{1}{4\pi} \int_S \hat{l}^\mu \;\; V \, \sigma_0' \, \bar{\sigma}_0' \, dS^2. \tag{17}$$

($\gamma_a{}^b{}_c$ is the tensor defined from the relation of the two covariant derivatives: $\nabla_{(B)a} \, j^b = \partial_a \, j^b + \gamma_a{}^b{}_c \, j^c$. Note that we are using abstract indices.)

The equation of motion in the null gauge. VI

The gravitational radiation degrees of freedom in the scalar V

Let us use the parameter γ to denote the order of the gravitational constant, and $\hat{\gamma}$ for some monotonic function of γ, to be determined later. The parameter $\hat{\gamma}$ can be thought as a measure of the effects of gravitational radiation.

We decompose the scalar V in terms of the following expression

$$V = V_\eta(1 + \hat{\gamma} V_{\hat{\gamma}}); \tag{18}$$

where V_η is the conformal factor of the angular part with respect to the flat background frame, and $V_{\hat{\gamma}}$ has information of first order in gravitational radiation.

Given a function $H(u, \zeta, \bar{\zeta})$ on the future null cones, one has the natural action of the Lorentz group on the angular coordinates [Goldberg et al.(1967)Goldberg, MacFarlane, Newman, Rohrlich, and Sudarshan, Held et al.(1970)Held, Newman, and Posadas], which allows us to make the decomposition

$$\bar{\eth}_{V_\eta} \eth_{V_\eta} H_l = -\frac{l(l+1)}{2} H_l; \tag{19}$$

for $l = 0, 1, 2, ...$; in terms of the edth operators of the instantaneous rest frame.

The equation of motion in the null gauge. VII

We use the decomposition for V in terms of eigenfunctions of the operator $\bar{\eth}_{V_\eta}\eth_{V_\eta}$:

$$V = V_\eta\Big(1 + \hat{\gamma} V_{\hat{\gamma}}\Big) = V_\eta\Big(1 + \hat{\gamma}(V_0 + V_1 + V_2 + V_3 + \ldots)\Big); \tag{20}$$

where we are omitting the subindex $_{\hat{\gamma}}$ on the right hand side, and instead we are using V_l, where the subindex l denotes the angular behavior.

Balanced equations of motion at higher order

Following an analysis in terms of the angular decomposition, we arrive at the equation of motion:

$$\boxed{\mathbf{a}^\mu = \mathbf{f}^\mu + \dot{\left(\alpha M \mathbf{f}^\nu\right)} + \beta M \mathbf{f}^\nu \mathbf{f}_\nu \mathbf{v}^\mu + f_\lambda^\mu;} \tag{21}$$

where $\dot{\left(\alpha M \mathbf{f}^\nu\right)}$ denotes the time derivative of $\left(\alpha M \mathbf{f}^\nu\right)$, and there are further equations that relate the parameter appearing in f_λ^μ; which we can not present in this occasion.

This improves the previous general form of the balanced equation of motion to a second order dynamics in the 'gravitational forces'.

Final comments. I

- From the tight relations between the interior and asymptotic structure in the null gauge model, we have arrived at the second order, in the accelerations, balanced equations of motion.
- The appearance of two new terms, at the second order in the accelerations, involving a quadratic term and a time derivative term; is common to the electromagnetic treatment of back reaction, and in other approaches of equations of motion.
- Our balanced equations of motion models does not impose restrictions for strong fields or high velocities.
- The field equations for the submetric can be solved to any desired order.
- Consider a binary system in which one mass of the particles, lets say B, is much bigger than the other. Then, our model, will assign the geometry for B, which will be a small perturbation of the Schwarzchild metric. The other particle would be governed by the equation of motion presented above.

  This, qualitative description for this thought binary system, is not produced by either the Post-Newtonian or the self-force models.

Thank you !

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