Representations of Weyl algebras

Vyacheslav Futorny

University of São Paulo, Brazil

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The n-th Weyl algebra $A_n = A_n(\mathbb{k})$:
generators: $x_i, \partial_i,\ i = 1, \ldots, n$
relations

$$x_i x_j = x_j x_i,\ \ \partial_i \partial_j = \partial_j \partial_i,$$
$$\partial_i x_j - x_j \partial_i = \delta_{ij},\ \ i, j = 1, \ldots, n.$$

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Let $P_n = \mathbb{C}[x_1, \ldots, x_n]$. $\forall \phi \in End(P_n)$ define $\phi_i = \phi(x_i)$.

Then ϕ defines a polynomial function

$$F : \mathbb{C}^n \to \mathbb{C}^n,$$

$F(z_1, \ldots, z_n) = (\phi_1(z_1), \ldots, \phi_n(z_n))$.
Let

$$J_F(\bar{x}) = (\partial f_i / \partial x_j),\ \Delta(F) = det(J_F).$$

If F is invertible then $\Delta(F) \in \mathbb{C}^*$.

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⋄ **Jacobian Conjecture**: Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function such that $\Delta(F) \in \mathbb{C}^*$. Then F is invertible (Keller, 1939).

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In A_n, the elements $t_i = \partial_i x_i$, $i = 1, \ldots, n$, generate the polynomial algebra $D = \mathbb{C}[t_i \mid i = 1, \ldots, n]$, which is a maximal commutative subalgebra of A_n. Denote by $\mathcal{G}$ the group generated by the automorphisms σ_i, $i = 1, \ldots, n$, of D, where $\sigma_i(t_j) = t_j - \delta_{i,j} a1$. Then $\mathcal{G}$ acts on the set $\mathfrak{max} D$ of maximal ideals of D.

A module VV for A_n is said to be a *weight module* if $VV = \bigoplus_{\mathfrak{m} \in \mathfrak{max} D} VV_{\mathfrak{m}}$, where $VV_{\mathfrak{m}} = \{v \in VV \mid \mathfrak{m} v = 0\}$.

If VV is a weight module and $VV_{\mathfrak{m}} \neq 0$ for $\mathfrak{m} \in \mathfrak{max} D$, then $\mathfrak{m}$ is said to be a *weight* of VV, and the set $\{\mathfrak{m} \in \mathfrak{max} D \mid VV_{\mathfrak{m}} \neq 0\}$ of weights of VV is the *support* of VV.

We have $x_i VV_{\mathfrak{m}} \subseteq VV_{\sigma_i(\mathfrak{m})}$ and $\partial_i VV_{\mathfrak{m}} \subseteq VV_{\sigma_i^{-1}(\mathfrak{m})}$.

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1. All indecomposable weight modules for A_n, $n < \infty$ were classified by Bavula, Bekkert (1999).

(i) Every irreducible weight module of $\mathcal{A}_1$ is isomorphic to one of the following $x_1^{\lambda_1}\mathbb{C}[x_1^{\pm 1}]$, $\lambda_1 \notin \mathbb{Z}$, $\mathbb{C}[x_1]$, $\mathbb{C}[x_1^{\pm 1}]/\mathbb{C}[x_1]$.

(ii) Every irreducible weight module of $\mathcal{A}_n = \bigotimes_{i=1}^{n} \mathcal{A}_{\{i\}}$ is isomorphic to the outer tensor product of n simple modules of $\mathcal{A}_{\{i\}}$, $i = 1, ..., n$.

2. All irreducible weight modules for A_∞ (with some restrictions) were classified by Benkart, Bekkert and V.F. (2000)

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For $\boldsymbol{\lambda} \in \mathbb{C}^{\mathbb{N}}$ we will consider formal expressions $\mathbf{x}^{\boldsymbol{\lambda}} = x_1^{\lambda_1} x_2^{\lambda_2} \ldots$ and assume that $\mathbf{x}^{\boldsymbol{\lambda}} \mathbf{x}^{\boldsymbol{\mu}} = \mathbf{x}^{\boldsymbol{\lambda}+\boldsymbol{\mu}}$. In particular, the standard basis of $\mathbb{C}[\mathbf{x}^{\pm 1}]$ can be written as $\{\mathbf{x}^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^{\mathbb{N}}_{\mathrm{fin}}\}$. For $\boldsymbol{\lambda} \in \mathbb{C}^{\mathbb{N}}$ define

$$\mathbf{x}^{\boldsymbol{\lambda}} \mathbb{C}[\mathbf{x}^{\pm 1}] := \{\mathbf{x}^{\boldsymbol{\lambda}} f \mid f \in \mathbb{C}[\mathbf{x}^{\pm 1}]\}.$$

(i) The module $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ is a weight module with basis consisting of the monomials $\mathbf{x}^{\boldsymbol{\lambda}+\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^{\mathbb{N}}_{\mathrm{fin}}$. In particular, $Supp(\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]) = \boldsymbol{\lambda} + \mathbb{Z}^{\mathbb{N}}_{\mathrm{fin}}$.

(ii) $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ is isomorphic to the $\mathcal{A}_\infty$-module with underlying vector space $\mathbb{C}[\mathbf{x}^{\pm 1}]$ and action

$$x_i \mapsto x_i; \partial_i \mapsto \partial_i + \lambda_i x_i^{-1}.$$

(iii) The $\mathcal{A}_\infty$-modules $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ and $\mathbf{x}^{\boldsymbol{\mu}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ are isomorphic if and only if $\boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathbb{Z}^{\mathbb{N}}_{\mathrm{fin}}$.

Let $\boldsymbol{\lambda} \in \mathbb{C}^{\mathbb{N}}$ be such that for every $i \in \mathbb{N}$, if $\lambda_i \in \mathbb{Z}$ then $\lambda_i \geq 0$. Then the $\mathcal{A}_\infty$-module $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ has a unique irreducible submodule

$$M = \{m \in \mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}] \mid \text{for every } i \in \mathcal{I}(\boldsymbol{\lambda}), \partial_i^n m = 0, \text{ for some } n > 0\},$$

where $\mathcal{I}(\boldsymbol{\lambda})$ stands for the set of all $i \in \mathbb{N}$ for which λ_i is integer.

The unique irreducible submodule M of $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ will be denoted by $\mathbf{x}^{\boldsymbol{\lambda}} C[\mathbf{x}^{\pm 1}]_\partial$.

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Example

Take $\boldsymbol{\lambda} = (1, 1, ...)$, i.e. $\lambda_i = 1$ for every i, and consider the simple module $M = \mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]_{\partial}$. Then M is ∂_i-locally finite, for every i on one hand, but there is no $m \in M$ for which $\partial_i m = 0$ for every i, on the other. Similar examples were considered by Mazorchuk, Zhao.

◇ The $\mathcal{A}_\infty$-module $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ is irreducible if and only if $\lambda_i \notin \mathbb{Z}$ for every $i \in \mathbb{N}$.

◇ If M is a irreducible weight $\mathcal{A}_\infty$-module with $\boldsymbol{\lambda} \in SuppM$ and such that all x_i, ∂_i act injectively on M, then $M \simeq \mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$.

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For $J \subset \mathbb{N}$ consider the multiplicative subset X_J of $\mathcal{A}_\infty$ generated by x_j, $j \in J$.

Denote by $\mathcal{D}_J\mathcal{A}_\infty$ the localization of $\mathcal{A}_\infty$ with respect to X_J.

For an $\mathcal{A}_\infty$-module M, let $\mathcal{D}_J M = \mathcal{D}_J\mathcal{A}_\infty \otimes_{\mathcal{A}_\infty} M$ the *J-localization of M*.

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Let $J \subset \mathbb{N}$, and $\boldsymbol{\mu} \in \mathbb{C}^{\mathbb{N}}$ be such that $\mu_i = 0$ if $i \notin J$. Define $\Psi_J^{\boldsymbol{\mu}} : \mathcal{D}_J\mathcal{A}_\infty \to \mathcal{D}_J\mathcal{A}_\infty$ to be the homomorphism

$$\begin{aligned} x_i &\mapsto x_i, i \in \mathbb{N}; \\ \partial_i &\mapsto \partial_i + \mu_i x_i^{-1}, i \in \mathbb{N}. \end{aligned}$$

Denote by $\Psi_J^{\boldsymbol{\mu}}(L)$ the $\mathcal{D}_J\mathcal{A}_\infty$-module obtained from L after twisting it by the automorphism $\Psi_J^{\boldsymbol{\mu}}$.

$\mathcal{D}_J^{\mu} M := \Psi_J^{\mu}(\mathcal{D}_J M)$ the (J, μ)-*twisted localization of* M.

(i) If $J = \mathbb{N}$ then $\mathcal{D}_J\mathbb{C}[\mathbf{x}] \simeq \mathbb{C}[\mathbf{x}^{\pm 1}]$.

(ii) Let $\boldsymbol{\lambda} \in \mathbb{C}^{\mathbb{N}}$ and $J \subset \mathbb{N}$ be such that $\lambda_i = 0$ for $i \notin J$. Then $\mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]$ is a $\mathcal{D}_J\mathcal{A}_\infty$-module which is isomorphic to $\Psi_J^{\boldsymbol{\lambda}}(\mathbb{C}[\mathbf{x}^{\pm 1}])$.

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Lemma

Let $J \subset \mathbb{N}$ and M be an irreducible weight $\mathcal{A}_\infty$-module on which x_i and ∂_j, $i \in \mathbb{N}, j \notin J$, act injectively, and ∂_j, $j \in J$, act nilpotently. Let $\boldsymbol{\mu} \in \mathbb{C}^{\mathbb{N}}$ be such that $\mu_i = 0$, $i \notin J$ and $\mu_j \notin \mathbb{Z}$, $j \in J$. Then $\mathcal{D}_J^{\boldsymbol{\mu}} M$ is an irreducible weight module on which all x_i, ∂_i, $i \in \mathbb{N}$ act injectively.

For any $J \subset \mathbb{N}$ we define the automorphism θ_J of $\mathcal{A}_\infty$ as follows.

$$\begin{aligned} \theta_J(x_j) = \partial_j, \theta_J(\partial_j) = -x_j, \quad &\text{if} \quad j \in J; \\ \theta_J(x_i) = x_i, \theta_J(\partial_i) = \partial_i, \quad &\text{if} \quad i \notin J. \end{aligned}$$

For an $\mathcal{A}_\infty$-module M, the module obtained from twisting M by θ_J will be denoted by M^{θ_J}.

Theorem

(i) Every irreducible weight module of $\mathcal{A}_\infty$ is isomorphic to $M(\boldsymbol{\lambda}, J) := \mathbf{x}^{\boldsymbol{\lambda}}\mathbb{C}[\mathbf{x}^{\pm 1}]_\partial^{\theta_J}$ for some $J \subset \mathcal{I}(\boldsymbol{\lambda})$ and some $\boldsymbol{\lambda} \in \mathbb{C}^{\mathbb{N}}$ with the property that $\lambda_i \in \mathbb{Z}$ implies $\lambda_i \geq 0$.

(ii) $M(\boldsymbol{\lambda}, J) \simeq M(\boldsymbol{\lambda}', J')$ if and only if $\boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}^{\mathbb{N}}_{\mathrm{fin}}$ and $J = J'$.

Cartan matrix $A = (a_{ij})$, $a_{ij} \leq 0$, $i \neq j$, $a_{ii} = 2$, $a_{ij} = 0 \Rightarrow a_{ji} = 0$ for all i, j, positive definite $\Rightarrow$

generators + Serre relations $\Rightarrow$ simple complex finite-dimensional Lie algebras.

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Example

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with the Lie bracket

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n}$$

(*loop* algebra).

◇ the universal central extension $\mathfrak{G} = \hat{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$,

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n(x, y)\delta_{n+m,0}c,$$

$d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ degree derivation, $d(x \otimes t^n) = n(x \otimes t^n)$, $d(c) = 0$ (*non-twisted affine* Kac-Moody algebra).

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$d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ degree derivation, $d(x \otimes t^n) = n(x \otimes t^n)$, $d(c) = 0$
(*non-twisted affine* Kac-Moody algebra).

A natural way to construct representations of affine Lie algebras is via induction from parabolic subalgebras. Induced modules play an important role in the classification problem of irreducible modules. For example, in the finite-dimensional setting any irreducible weight module is a quotient of the module induced from an irreducible module over a parabolic subalgebra, and this module is *dense* (that is, it has the largest possible set of weights) as a module over the Levi subalgebra of the parabolic (Fernando, V.F.). Dense irreducible module is always torsion free if all weight spaces are finite-dimensional. In the affine case, a similar conjecture singles out induced modules as construction devices for irreducible weight modules. This conjecture has been shown for $A_1^{(1)}$ (V.F.), $A_2^{(2)}$ (Bunke) and for all affine Lie algebras for modules with finite-dimensional weight spaces and nonzero central charge (V.F., Tsylke).

In the latter case, a phenomenon of reduction to modules over a proper subalgebra (finite-dimensional reductive) provides a classification of irreducible modules. Moreover, recent results of Dimitrov and Grantcharov show the validity of the conjecture also for modules with finite-dimensional weight spaces and zero central charge.

A closed subset $P \subset \Delta$ is called a *partition* if $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$ (root system). In the case of finite-dimensional simple Lie algebras, every partition corresponds to a choice of positive roots in Δ, and all partitions are conjugate by the Weyl group. The situation is different in the infinite-dimensional case. In the case of affine Lie algebras the partitions are divided into a finite number of Weyl group orbits (Jakobsen-Kac, V.F.).

Given a partition P of Δ, we define a *Borel* subalgebra $B_P \subset \mathfrak{G}$ generated by $\mathfrak{H}$ and the root spaces $\mathfrak{G}_\alpha$ with $\alpha \in P$. Hence, in the affine case not all of the Borel subalgebras are conjugate but there exists a finite number of conjugacy classes.

A *parabolic* subalgebra of $\mathfrak{G}$ corresponds to a *parabolic* subset $P \subset \Delta$, which is a closed subset in Δ such that $P \cup (-P) = \Delta$. Given such a parabolic subset P, the corresponding parabolic subalgebra $\mathfrak{G}_P$ is generated by $\mathfrak{H}$ and all the root spaces $\mathfrak{G}_\alpha$, $\alpha \in P$.

The conjugacy classes of Borel subalgebras of $\mathfrak{G}$ are parameterized by the parabolic subalgebras of the associated finite-dimensional simple Lie algebra $\mathfrak{g}$.

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The conjugacy classes of Borel subalgebras of $\mathfrak{G}$ are parameterized by the parabolic subalgebras of the associated finite-dimensional simple Lie algebra $\mathfrak{g}$.

Let P be a parabolic subset of Δ. Let N be an irreducible weight module over $\mathfrak{P} = \mathfrak{G}_P = \mathfrak{P}_0 \oplus \mathfrak{P}^+$, with a trivial action of $\mathfrak{P}^+$, and

$$M_P(N) = \mathrm{ind}(\mathfrak{P}, \mathfrak{G}; N).$$

Then $M_P(N)$ has a unique irreducible quotient $L_P(N)$.

Borel subalgebras $\rightsquigarrow$ Verma modules/Verma type modules

Parabolic subalgebras $\rightsquigarrow$ generalized Verma modules/generalized Verma type modules

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The subspace L $:= \mathbb{C}c \oplus \bigoplus_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{G}_{n\delta}$ forms a Heisenberg Lie subalgebra of the affine algebra $\mathfrak{G}$.

Theorem (Benkart, Bekkert, V.F., Kashuba, 2012; V.F., Kashuba, $\geq$ 2013)

Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) \neq 0$, and assume VV is an irreducible L-module where c acts by $\lambda(c)$. Then $ind(L, \mathfrak{G}; VV)$ is an irreducible $\mathfrak{G}$-module.

OBRIGADO