Representations of Copointed Hopf Algebras

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Abstract

In this paper we investigate a family of copointed Hopf algebras of the Nichols algebra of the affine rack $(\mathcal{F}_4, \omega)$.

This is a joint work with Nicolás Andruskiewitsch and Cristian Vay.

Conventions

$\Bbbk$ is an algebraically closed field of characteristic zero;

If G is a group, we denote by $\Bbbk G$ the group algebra of G and by $\Bbbk^G$ the function algebra of G.

The usual basis of $\Bbbk G$ is $\{g : g \in G\}$ and $\{\delta_g : g \in G\}$ is its dual basis of $\Bbbk^G$, i.e., $\delta_g(h) = \delta_{g,h}$ for every $g, h \in G$.

Conventions

If H is a Hopf algebra, then Δ, ε, $\mathcal{S}$ denote respectively the comultiplication, the counit and the antipode.

Let ${}^H_H\mathcal{YD}$ be the category of Yetter-Drinfeld module over H.

The Nichols algebra $\mathcal{B}(V)$ of $V \in {}^H_H\mathcal{YD}$ is the graded quotient $T(V)/\mathcal{J}$ where $\mathcal{J} = \bigoplus_{l \geq 2} \mathcal{J}^l$ is the largest Hopf ideal of $T(V)$ generated by homogeneous elements of degree ≥ 2.

$(\mathcal{F}_4, \omega)$

Let $\mathcal{F}_4$ be the finite field of four elements and $\omega \in \mathcal{F}_4$ such that $\omega^2 + \omega + 1 = 0$. The affine rack $(\mathcal{F}_4, \omega)$ is the set $\mathcal{F}_4$ with operation $a \triangleright b = \omega b + \omega^2 a$.

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Let $(\cdot, g, \chi_G)$ be a *faithful principal YD-realization* of $((\mathcal{F}_4, \omega), -1)$ over a finite group G, that is

- $\cdot$ is an action of G on $\mathcal{F}_4$,
- $g : \mathcal{F}_4 \to G$ is an injective function such that $g_{h \cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \triangleright j$ for all $i, j \in \mathcal{F}_4$,
- $\chi_G : G \to \mathbb{k}^*$ is a multiplicative character such that $\chi_G(g_i) = -1$ for all $i \in \mathcal{F}_4$.

$V(\mathcal{F}_4, \omega)$

These data define a structure on $V(\mathcal{F}_4, \omega) = \Bbbk\{x_i\}_{i \in \mathcal{F}_4}$ of Yetter-Drinfeld module over $\Bbbk^G$ via

$$\delta_t \cdot x_i = \delta_{t, g_i^{-1}} x_i \quad \text{and} \quad \lambda(x_i) = \sum_{t \in G} \chi_i(t^{-1}) \delta_t \otimes x_{t^{-1} \cdot i} \,.$$

$\mathcal{B}(\mathcal{F}_4, \omega)$

$\mathcal{B}(\mathcal{F}_4, \omega)$ is the quotient of $T(V) = T(V(\mathcal{F}_4, \omega))$ by the ideal $\mathcal{J}(\mathcal{F}_4, \omega)$ generated by

$$x_i^2,$$

$$x_j\, x_i + x_i\, x_{(\omega+1)i+\omega j} + x_{(\omega+1)i+\omega j}\, x_j \quad \forall i, j \in \mathcal{F}_4 \text{ and}$$

$$z := (x_\omega x_0 x_1)^2 + (x_1 x_\omega x_0)^2 + (x_0 x_1 x_\omega)^2.$$

To obtain a basis of $\mathcal{B}(\mathcal{F}_4, \omega)$, which we will denote by $\mathbb{B}$, we choose one element per row of the next list and multiply them from top to bottom:

$$1, x_0,$$

$$1, x_1, \; x_1 x_0,$$

$$1, x_\omega x_0 x_1,$$

$$1, x_\omega, \; x_\omega x_0,$$

$$1, x_{\omega^2}.$$

$\mathcal{A}_{G,\lambda}$

Let $\lambda \in \Bbbk$ and assume $z \in T(V)[e]$. The Hopf algebra $\mathcal{A}_{G,\lambda}$ is the quotient of $T(V)\#\Bbbk^G$ by the ideal generated by

$$x_i^2,$$

$$x_j\, x_i + x_i\, x_{(\omega+1)i+\omega j} + x_{(\omega+1)i+\omega j}\, x_j \quad \forall i,j \in \mathcal{F}_4 \text{ and}$$

$$\text{z-f} \quad \text{where} \quad f = \lambda(1-\chi_z^{-1}) \quad \text{and} \quad \chi_z = \chi_G^6.$$

Notice that if either $\lambda = 0$ or $\chi_z = 1$, then $\mathcal{A}_{G,\lambda} = \mathcal{B}(\mathcal{F}_4,\omega)\#\Bbbk^G$.

$\mathcal{A}_{G,\lambda}$

We think on $\mathcal{A}_{G,\lambda}$ as an algebra with generators $\{x_i, \delta_g : i \in \mathcal{F}_4,\ g \in G\}$ with relations:

$$\delta_g x_i = x_i \delta_{g_i g}, \quad x_i^2 = 0, \quad \delta_g \delta_h = \delta_g(h)\delta_g, \quad 1 = \sum_{g \in G} \delta_g,$$

$$x_0 x_\omega + x_\omega x_1 + x_1 x_0 = 0 = x_0 x_{\omega^2} + x_{\omega^2} x_\omega + x_\omega x_0,$$

$$x_1 x_{\omega^2} + x_0 x_1 + x_{\omega^2} x_0 = 0 = x_\omega x_{\omega^2} + x_1 x_\omega + x_{\omega^2} x_1 \quad \text{and}$$

$$x_\omega x_0 x_1 x_\omega x_0 x_1 + x_1 x_\omega x_0 x_1 x_\omega x_0 + x_0 x_1 x_\omega x_0 x_1 x_\omega = \lambda(1 - \chi_z^{-1}),$$

for all $i \in \mathcal{F}_4$ and $g \in G$.

A basis for $\mathcal{A}_{G,\lambda}$ is $\mathbb{A} = \{x\delta_g | x \in \mathbb{B}, g \in G\}$.

Theorem

Let H be a lifting Hopf algebra of $\mathcal{B}(\mathcal{F}_4, \omega)$ over $\Bbbk^G$. Then

- (a) *If $z \in T(V)^\times$, then $H \simeq \mathcal{B}(\mathcal{F}_4, \omega)\#\Bbbk^G$.*
- (b) *If $z \in T(V)[e]$, then $H \simeq \mathcal{A}_{G,\lambda}$ for some $\lambda \in \Bbbk$.*
- (c) *$\mathcal{A}_{G,\lambda}$ is a cocycle deformation of $\mathcal{A}_{G,\lambda'}$, for all $\lambda, \lambda' \in \Bbbk$.*
- (d) *$\mathcal{A}_{G,\lambda}$ is a lifting of $\mathcal{B}(\mathcal{F}_4, \omega)$ over $\Bbbk^G$ for all $\lambda, \lambda' \in \Bbbk$.*
- (e) *$\mathcal{A}_{G,\lambda} \simeq \mathcal{A}_{G,1} \not\simeq \mathcal{A}_{G,0}$ for all $\lambda \in \Bbbk^*$.* □

Case 1

If either $\lambda = 0$ or $\chi_z = 1$, then $\mathcal{A}_{G,\lambda} = \mathcal{B}(\mathcal{F}_4, \omega)\#\Bbbk^G$. In this case, the simple modules of $\mathcal{B}(\mathcal{F}_4, \omega)\#\Bbbk^G$ are $\{\Bbbk_h\}_{h\in G}$, where $\Bbbk_g$ are one-dimensional $\Bbbk^G$-modules of weight g.

Case 2

From now we fix $\lambda \in \mathbb{k}^*$ and assume $z \in T(V)[e]$ and $\chi_z \neq 1$. For $g \in G \setminus \ker \chi_z$, we define

$$e_1^g = -\frac{1}{f(g)} b_1 \delta_g, \qquad e_2^g = -\frac{1}{f(g)} b_2 \delta_g, \qquad e_3^g = \frac{1}{f(g)} b_3 \delta_g,$$

$$e_4^g = \frac{1}{f(g)} (b_4 - b_3) \delta_g, \qquad e_5^g = \frac{1}{f(g)} (b_5 + b_1) \delta_g \qquad \text{and}$$

$$e_6^g = \delta_g + \frac{1}{f(g)} (b_2 - b_4 - b_5) \delta_g,$$

where

$$b_1 = x_0 x_1 x_0 x_\omega x_0 x_{\omega^2}, \quad b_2 = x_0 x_\omega x_0 x_1 x_\omega x_{\omega^2}, \quad b_3 = x_1 x_0 x_\omega x_0 x_1 x_{\omega^2}$$

$$b_4 = x_1 x_\omega x_0 x_1 x_\omega x_0, \qquad b_5 = x_0 x_1 x_\omega x_0 x_1 x_\omega.$$

Case 2

A complete set of orthogonal primitive idempotents of $\mathcal{A}_{G,\lambda}$ is

$$\mathcal{E} := \left\{ \delta_h,\, e_1^g,\, e_2^g,\, e_3^g,\, e_4^g,\, e_5^g,\, e_6^g \mid h \in \ker \chi_z,\, g \in G \setminus \ker \chi_z \right\}.$$

Theorem

Let $e_i^g \in \mathcal{E}$. Then $\mathcal{A}_{G,\lambda} e_i^g$ is an injective and projective simple module of dimension 12.

Referências

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