Permutads

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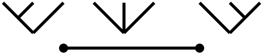
Planar rooted trees

Associahedron $\mathfrak{A}_n$ is a *n-1* dimensional polytope, whose faces of dimension r correspond to planar rooted trees with *n+1* leaves and *n-r* internal vertices.
Let $\mathcal{T}_n^r$ denotes the set of planar rooted trees with *n+1* leaves and r internal vertices,

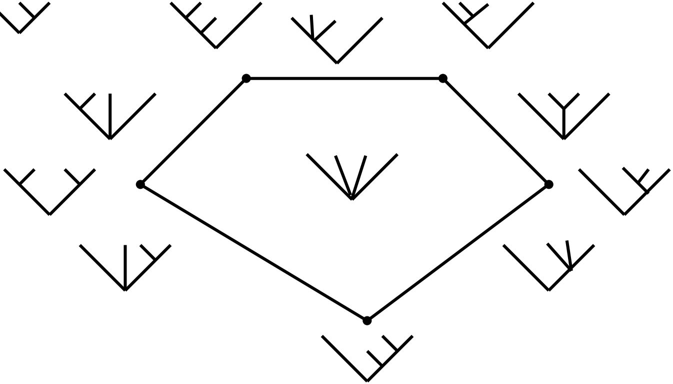
$\mathcal{T}_0^0 = \{$ $\}$ $\mathcal{T}_1^1 = \{$  $\}$ $\mathcal{T}_2^2 = \{$   $\}$

$\mathcal{T}_2^1 = \{$  $\}$

Associahedron or Stasheff polytope



$\mathfrak{A}_2$



$\mathfrak{A}_3$

Algebraic structures associated to the Stasheff polytope

- Pre-Lie system is a colored operad (i.e. the operations are not always defined, they depend on the degree of the elements). Pre-Lie systems are equivalent to non-Σ operads, modulo a shift of the degree. We add a section describing non-Σ operads as monads in certain category of functors, following V. Ginzburg and M. Kapranov).
- Tridendriform is a non-Σ operad, which extend the notion of dendriform algebras defined by J.-L. Loday .

Pre-Lie systems

Graftings

A pre-Lie system or non-symmetric operad is a graded vector space $L = \bigoplus_{n \geq 0} L_n$ equipped with linear maps

$$\circ_i : L_m \otimes L \longrightarrow L, \text{ for } 1 \leq i \leq m,$$

satisfying

1. $x \circ_j (y \circ_i z) = (x \circ_j y) \circ_{i+j} z$, for $0 \leq i \leq | y |$ and $0 \leq j \leq | x |$,
2. $(x \circ_j y) \circ_i z = (x \circ_i z) \circ_{j+|z|} y$, for $0 \leq i < j$.

Remark: Let $(L, \circ_i)$ be a pre-Lie system, then L with the binary product

$$x \circ y := \sum_{i=0}^{|x|} x \circ_i y,$$

is a pre-Lie algebra, as defined by M. Gerstenhaber.

Gerstenhaber's example: The space of Hochschild cochains of an associative algebra A.

Free pre-Lie systems

Let t and w be planar rooted trees, the element $t \circ_i w$ is the tree obtained by grafting the root of w on the i-th. leaf of t.

Free objects Denote by Pre-Lie(V) the free pre-Lie system spanned by a vector space V. The vector space spanned by all planar binary rooted trees, with the $\circ_i$'s, is the free pre-Lie system spanned by one element Pre-Lie($\mathbb{K}$). The vector space spanned by all planar rooted trees, with the operations $\circ_i$, is the free pre-Lie system spanned by the graded vector space $\bigoplus_{n\geq 1} \mathbb{K}c_n$.

In general, let X be a basis of a vector space V. Let $\mathcal{T}_n^X$ be the set of planar rooted trees with $n+1$ leaves and the vertices colored by the elements of X in such a way that each vertex with $r+1$ inputs is colored by an element of X of degree r.
The free pre-Lie system spanned by V is the space spanned by the set $\bigcup_n \mathcal{T}_n^X$ with the product $\circ_i$ given by the grafting at the i-th. leaf.

Relation with non-Σ-operads

Let L be a pre-Lie system, define $\mathcal{P}$ as $\mathcal{P}_{n+1} = L_n$, with the partial operation $\circ_i$ and the trivial action of the symmetric group Σ_{n+1}, then $\mathcal{P}$ is non-Σ operad.

Coalgebra structure on a pre-Lie system

Let $L = \bigoplus_{n \geq 0} L_n$ be a pre-Lie system. A coproduct on L is a linear map $\Delta : L \longrightarrow L \otimes L$ such that:

$$\Delta(x \circ_i y) = \sum_{|x_{(1)}|<i} x_{(1)} \otimes (x_{(2)} \circ_{i-|x_{(1)}|} y) +$$

$$\sum_{|x_{(1)}|=i} (x_{(1)} \circ_i y_{(1)}) \otimes (x_{(2)} \circ_0 y_{(2)}) + \sum_{|x_{(1)}|>i} (x_{(1)} \circ_i y) \otimes x_{(2)}.$$

If (V, Θ) is a graded coassociative coalgebra, then Pre-Lie(V) is equipped with a natural coproduct:

1. $\Delta_\Theta(c_n, x) := \sum_{i=0}^{n} \sum_{|x_{(1)}=i|}(c_i, x_{(1)}) \otimes (c_{n-i}, x_{(2)})$.
2. Δ_Θ is a coproduct for the pre-Lie system structure of Pre-Lie(V).

Coproduct and pre-Lie structure

Let $(L, \circ_i)$ be a pre-Lie system. The products:

- $x \circ_0 y$,
- $x \circ_L y = x \circ_{|x|} y$,

are associative.

Remark: Note that the relationships satisfied by Δ and the $\circ_i$'s imply that:

1.

$$\Delta(x \circ y) = \sum x_{(1)} \otimes (x_{(2)} \circ y) + (x_{(1)} \circ y) \otimes x_{(2)} + \sum (x_{(1)} \circ_L y_{(1)}) \otimes (x_{(2)} \circ_0 y_{(2)}).$$

2. $\Delta(x \circ_0 y) = \sum y_{(1)} \otimes (x \circ_0 y_{(2)})$,
3. $\Delta(x \circ_L y) = \sum x_{(1)} \otimes (x_{(2)} \circ_L y) + (x \circ_L y_{(1)}) \otimes y_{(2)}$.

$(L, \circ_0^{op}, \Delta)$ and $(L, \circ_L^{op}, \Delta)$ are unital infinitesimal algebras.

Bar construction

Let (V, θ) be a conilpotent coassociative coalgebra. Define a boundary map on Pre-Lie(V) as follows:

1. $\delta(x) := \sum (-1)^{i|x_{(1)}|} x_{(1)} \circ_i x_{(2)}$, for $x \varepsilon V$ with $\overline{\theta}(x) = \sum x_{(1)} \otimes x_{(2)}$,
2. δ is a derivation for all the binary products $\circ_i$.

When $V = \bigoplus \mathbb{K} c_n$ is the space spanned by all corollas with the coproduct given by:

$$\theta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i},$$

we get the associahedra as the bar construction.

Non-Σ operads as monads (V. Ginzburg and M. Kapranov)

Let $\mathbb{N}^+$Vect be the category of positively graded vector spaces. Objects of $\mathbb{N}^+$Vect are families $M = \{M_n\}$ of $\mathbb{K}$-vector spaces. Planar rooted trees define a monad in the category $\mathbb{N}^+$Vect as follows:

- For $t \,\varepsilon\, \mathcal{T}_n$,

$$M_t := \bigotimes_{v \,\varepsilon\, Vert(t)} M_{|v|},$$

  $\mid v \mid$ is the number of inputs of v.

- For $M \,\varepsilon\, \mathbb{N}^+$Vect, the graded vector space $\mathbb{P}(M)$ is:

$$\mathbb{P}(M) := \bigoplus_{t \,\varepsilon\, \mathcal{T}_n} M_t,$$

  and $\mathbb{P}(M)_1 := \mathbb{K}$.

The map $\iota(M) : M \longrightarrow \mathbb{P}(M)$ consist in associating to $\mu \,\varepsilon\, M_n$ the corolla with $n+1$ leaves, colored by μ.
$\mathbb{P}_{n+1}(M)$ is spanned by planar rooted trees with $n+1$ leaves and the vertices decorated by elements of M.
The grafting of trees defines $\Gamma : \mathbb{P} \circ \mathbb{P} \longrightarrow \mathbb{P}$

$$\Gamma(t; w_0, \dots, w_n) := (((t \circ_0 w_0) \circ_{|w_0|+1} w_1) \dots) \circ_{|w_0|+\dots+|w_{n-1}|+1} w_n,$$

which is an associative and unital transformation of functors.

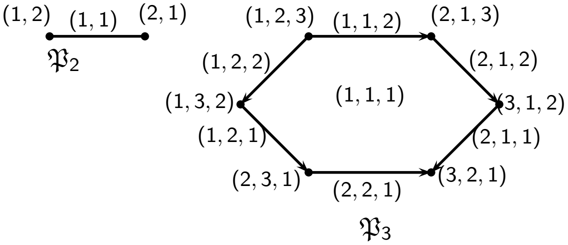
$(\mathbb{P}, \Gamma, \iota)$ is a monad in the category $\mathbb{N}^+$Vect.
A non-Σ operad is an algebra over this monad. That is, an object $M = \{M(n)\}_{n\geq 1}$ in $\mathbb{N}^+$Vect with:

$$\mathbb{P}(M) \longrightarrow M,$$

compatible with Γ and ι.

Permutohedra

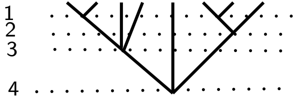
The permutohedron $\mathfrak{P}_n$ is a *n-1* dimensional polytope whose faces of dimension r correspond to all surjective maps from $\{1, \ldots, n\}$ to $\{1, \ldots, n-r\}$.
Surj_n is the set of surjective maps defined on $\{1, \ldots, n\}$.
Note that for $r = 0$, we get the set Σ_n of permutations of n elements.


(1, 2)
(1, 1)
(2, 1)
$\mathfrak{P}_2$
(1, 2, 3)
(1, 1, 2)
(2, 1, 3)
(1, 2, 2)
(2, 1, 2)
(1, 3, 2)
(1, 1, 1)
(3, 1, 2)
(1, 2, 1)
(2, 1, 1)
(2, 3, 1)
(3, 2, 1)
(2, 2, 1)
$\mathfrak{P}_3$

Relation with associahedra

There exist canonical maps $\mathrm{Surj}_n \longrightarrow \mathcal{T}_n$ which are graded and surjective.

Idea:



$\longmapsto (1, 3, 3, 4, 4, 1, 2)$

A. Tonks: The associahedron may be obtained from the permutohedron by contracting some faces.

Shuffle algebras

A shuffle algebra is a graded vector space $A = \bigoplus_{n \geq 1} A_n$, endowed with operations:

$$\bullet_\gamma : A_m \otimes A_n \longrightarrow A_{n+m}, \text{ for } \gamma \varepsilon \mathrm{Sh}(n,m)$$

satisfying:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $\gamma \cdot (\delta \times 1_n) = \lambda \cdot (1_r \times \sigma)$ in $\mathrm{Sh}(n,m,r)$.
Since any k-shuffle $\sigma \varepsilon \mathrm{Sh}(i_1, \dots, i_k)$ can be written as a composition of 2-shuffles, there exists:

$$\bullet_\sigma : A_{i_1} \otimes \cdots \otimes A_{i_k} \longrightarrow A_{i_1 + \cdots + i_k}.$$

Free shuffle algebras

The vector space spanned by all permutations, with the operations:

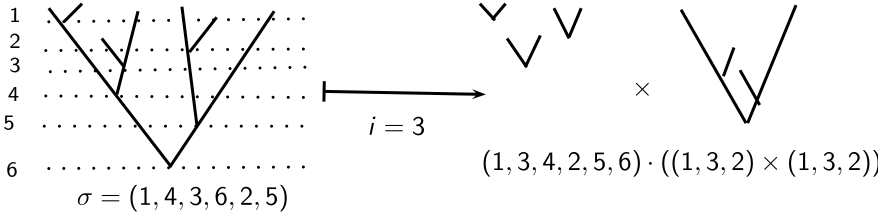
$$\alpha \bullet_\gamma \beta := (\beta \times \alpha) \cdot \gamma^{-1},$$

is the free shuffle algebra spanned by one element.
The vector space spanned by $\bigcup_{n \geq 1} \mathrm{Surj}_n$ is the free shuffle algebra spanned by the maps $c_n = (1, \ldots, 1) : \{1, \ldots, n\} \longrightarrow \{1\}$.

Coalgebra structure

Let $\sigma \,\varepsilon\, \Sigma_n$ and $0 \leq i \leq n$, there exist unique $\gamma \,\varepsilon\, \mathrm{Sh}(i, n-i)$, $\sigma^i_{(1)} \,\varepsilon\, \Sigma_i$ and $\sigma^i_{(2)} \,\varepsilon\, \Sigma_{n-i}$ such that

$$\sigma = \gamma \cdot (\sigma^i_{(1)} \times \sigma^i_{(2)}).$$



A shuffle bialgebra is a shuffle algebra A equipped with a coassociative coproduct Δ such that:

$$\Delta(x \bullet_{\sigma} y) = \sum_{r=1}^{n+m-1} \Big(\sum (x_{(1)} \bullet_{\sigma^{r}_{(1)}} y_{(1)}) \otimes (x_{(2)} \bullet_{\sigma^{n+m-r}_{(2)}} y_{(2)}) \Big).$$

Remark: A pre-Lie system, equipped with a coproduct is a shuffle bialgebra.

Bar construction

Let (V, θ) be a conilpotent coassociative coalgebra. Define a boundary map on Shuff(V) as follows:

1. $\delta(x) := \sum \operatorname{sgn}(\sigma)(-1)^{|x_{(1)}|} x_{(1)} \bullet_\sigma x_{(2)}$, for $x \varepsilon V$ with $\overline{\theta}(x) = \sum x_{(1)} \otimes x_{(2)}$,
2. δ is a derivation for all the binary products $\bullet_\sigma$.

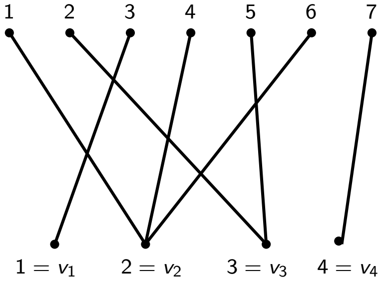
When $V = \bigoplus \mathbb{K} c_n$ is the space spanned by all corollas with the coproduct given by:

$$\theta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i},$$

we get the permutohedra as the bar construction.

Arity

Let $\mathrm{Sur}(\underline{n},\underline{k})$ denotes the set of a surjective maps from $\{1,\ldots,n\}$ to $\{1,\ldots,k\}$.
A vertex of $t \varepsilon \mathrm{Sur}(\underline{n},\underline{k})$ is an element in the image of t.
The arity of $v \varepsilon \mathrm{Vert}(t)$ is $\mid v \mid := \# t^{-1}(v) + 1$.



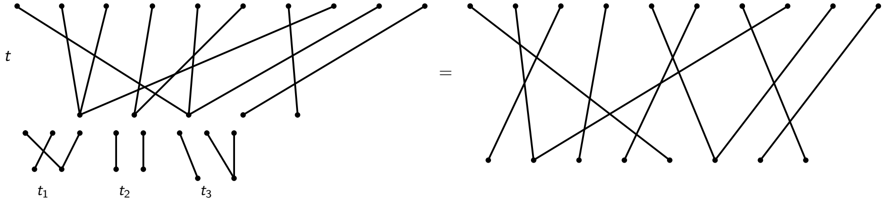
$f = (2,3,1,2,3,2,4)$

Substitution

Let $t \,\varepsilon\, \mathrm{Sur}(\underline{n}, \underline{k})$ and $t_j \,\varepsilon\, \mathrm{Sur}(\underline{i_j}, \underline{m_j})$, $j = 1, \ldots, k$, be surjective maps such that $i_j = \# t^{-1}(j)$. Let $m := \sum_j m_j$.
The substitution of $\{t_j\}$ in t is the surjective map $(t; t_1, \ldots, t_k) \,\varepsilon\, \mathrm{Sur}(\underline{n}, \underline{m})$ given by

$$(t; t_1, \ldots, t_k)(a) := m_1 + \cdots + m_{j-1} + t_j(b),$$

whenever $t(a) = j$ and a is the b-th element in $t^{-1}(j)$.



Substitution is associative.

Monad

(joint paper with J.-L. Loday, to appear in J. of Combinatorial Theory, Series A)

Surjective maps define a monad in the category $\mathbb{N}^+$Vect as follows:

- For $t \,\varepsilon\, \mathrm{Sur}(\underline{n}, \underline{k})$,

$$M_t := \bigotimes_{v \,\varepsilon\, Vert(t)} M_{|v|},$$

  $|\, v \,|$ is the number of inputs of v.

- For $M \,\varepsilon\, \mathbb{N}^+$Vect, the graded vector space $\mathbb{P}(M)$ is:

$$\mathbb{P}(M)_{n+1} := \bigoplus_{t \,\varepsilon\, Surj_n} M_t,$$

  and $\mathbb{P}(M)_1 := \mathbb{K}$.

Permutad

Result: The substitution of surjective maps defines a transformation of functors $\Gamma : \mathbb{P} \circ \mathbb{P} \longrightarrow \mathbb{P}$ which is associative and unital. So $(\mathbb{P}, \Gamma, \iota)$ is a monad on graded vector spaces.

A permutad is a unital algebra over the monad $(\mathbb{P}, \Gamma, \iota)$.

Applications

1. Study of combinatorial Hopf algebras (Hivert-Novelli-Thibon, Aguiar-Sottile, Lam-Pylyavskyy,...).
2. Generalized associahedra (Carr, Devadoss, Forcey)
3. Shuffle operads (Dotsenko, Koroshkin)