Root Multiplicities of the Kac-Moody Algebra $HD_n^{(1)}$

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Outline

Classification of Kac-Moody Algebras

Symmetrizable Kac-Moody algebras can be classified into three types, based on the properties of their GCM:

1. finite,
2. affine,
3. indefinite.

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Some Motivation

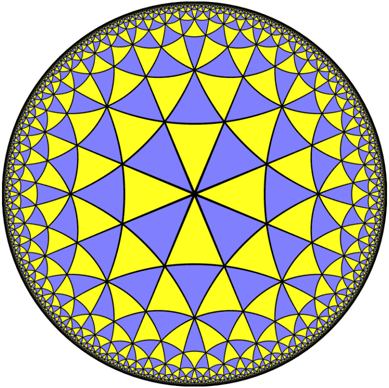

Figure: Example of a hyperbolic tesselation.

Root Multiplicities

An element $\alpha \in \mathfrak{h}^*$, is called a root of $\mathfrak{g}$ if $\alpha \neq 0$ and $\{0\} \neq \mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x, h \in \mathfrak{h}\}$.

A fundamental problem in the study of Kac-Moody algebras is to determine the root multiplicities, i.e. the dimension of the root spaces $\mathfrak{g}_\alpha$.

These are known to be 1 for finite type, and affine type root multiplicities are also all known.

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For indefinite type Kac-Moody algebras, some results have been given for

1. $HA_1^{(1)}$ (Feingold and Frankel, 1983)
2. $HA_n^{(1)}$ (Kang and Melville, 1994),
3. $E_{10} = HE_8^{(1)}$ (Kac, Moody, Wakimoto, 1998), $HG_2^{(1)}$ and $HD_4^{(3)}$ (Hontz and Misra, 2002), and
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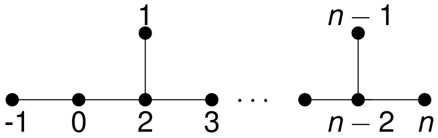
Introducing $HD_n^{(1)}$

Consider the Kac-Moody algebra $HD_n^{(1)}$ with Dynkin diagram given below:



It is an indefinite Kac-Moody algebra containing the affine Kac-Moody algebra $D_n^{(1)}$. There is a construction of $HD_n^{(1)}$ in terms of $D_n^{(1)}$-modules.

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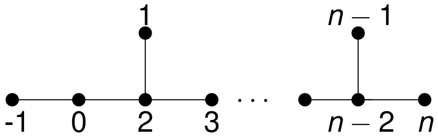

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Outline of the Construction

The construction of $HD_n^{(1)}$ has three ingredients:

- The Lie algebra $\mathfrak{g}_0 := D_n^{(1)}$.
- The $\mathfrak{g}_0$-modules $V(\Lambda_0)$ and $V^*(\Lambda_0)$.
- A $\mathfrak{g}_0$-module homomorphism $\psi : V^*(\Lambda_0) \otimes V(\Lambda_0) \to \mathfrak{g}_0$.

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The $\mathfrak{g}_0$-module homomorphism.

Theorem ([Benkart Kang Misra 93])

The map

$$\psi : V^*(\Lambda_0) \otimes V(\Lambda_0) \to \mathfrak{g}_0$$

given by

$$v^* \otimes w \mapsto -\sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot w \rangle x_i - 2\langle v^*, w \rangle c$$

is a $\mathfrak{g}_0$-module homomorphism, where $\mathfrak{g}_0$ is considered as a module under the adjoint action.

The Lie Algebra $\tilde{\mathfrak{g}}$.

Now, let $\tilde{\mathfrak{g}}_1 := V(\Lambda_0)$, $\tilde{\mathfrak{g}}_{-1} := V^*(\Lambda_0)$, $\tilde{\mathfrak{g}}_-$ and $\tilde{\mathfrak{g}}_+$ be the free Lie algebras generated by $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_1$ respectively. Let $\tilde{\mathfrak{g}}_{\pm i} = \text{span}\{[y_1, [y_2, [\ldots, [y_{i-1}, y_i] \ldots]]] | y_1, y_2, \ldots, y_i \in \tilde{\mathfrak{g}}_{\pm 1}\}$.

Theorem ([Benkart Kang Misra 93])

The vector space $\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}}_- \oplus \mathfrak{g}_0 \oplus \tilde{\mathfrak{g}}_+$ is a graded Lie algebra with bracket given by extending the following:

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The Ideal J.

Let $J_{\pm k} := \{x \in \tilde{\mathfrak{g}}_k | [y_1, [y_2, \ldots, [y_{k-1}, x] \ldots]] = 0, \forall y_1, y_2, \ldots, y_{k-1} \in \tilde{\mathfrak{g}}_{\pm 1}\}$. Let $J_{\pm} := \bigoplus_{k>1} J_{\pm k}$ and $J := J_+ \oplus J_-$.

Theorem ([Benkart Kang Misra 93])

J_+ and J_- are ideals of $\tilde{\mathfrak{g}}$, and J is the maximal graded ideal that intersects $\tilde{\mathfrak{g}}_{-1} \oplus \mathfrak{g}_0 \oplus \tilde{\mathfrak{g}}_1$ trivially.

Theorem ([Benkart Kang Misra 93])

Let $\mathfrak{g}(A)$ be the Kac-Moody algebra with GCM A, and let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}(A)$. Then $\mathfrak{g}(A)$ is isomorphic to $\tilde{\mathfrak{g}}/J$.

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Kang's Multiplicity Formula

By Kostant's theorem for Lie algebra cohomology and the Euler-Poincaré principle:

$$\prod_{\alpha\in\Delta^-(S)} (1-e(\alpha))^{\dim(\mathfrak{g}_\alpha)} = 1 - \sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}} (-1)^{\ell(w)+1}\mathrm{ch}(V(w(\rho)-\rho))$$

where

- $S=\{0,1,\ldots,n\}$: The index set of simple roots of $\mathfrak{g}_0 = D_n^{(1)}$.
- Δ_S : The set of roots of $\mathfrak{g}_0$.
- $\Delta_S^{\pm}$: The set of positive (resp. negative) roots of $\mathfrak{g}_0$.
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Perfect Crystals

Definition (Perfect Crystal)

For a positive integer $l > 0$, we say that a finite classical crystal $\mathcal{B}$ is a *perfect crystal of level l* if it satisfies the following conditions:

1. there exists a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal base whose crystal graph is isomorphic to $\mathcal{B}$,
2. $\mathcal{B} \otimes \mathcal{B}$ is connected,
3. there exists a classical weight $\lambda_0 \in \bar{P}$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i$, and $\#(\mathcal{B}_{\lambda_0}) = 1$,

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Perfect Crystal (cont.)

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4. for any $b \in \mathcal{B}$, we have $\langle c, \varepsilon(b) \rangle \geq l$,
5. for each $\lambda \in \bar{P}_l$ there exist unique $b^\lambda \in \mathcal{B}$ and $b_\lambda \in \mathcal{B}$ such that $\varepsilon(b^\lambda) = \lambda, \varphi(b_\lambda) = \lambda$,

where $\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$, and $\bar{P}_l = \{\lambda \in \bar{P}^+ | \langle c, \lambda \rangle = l\}$.

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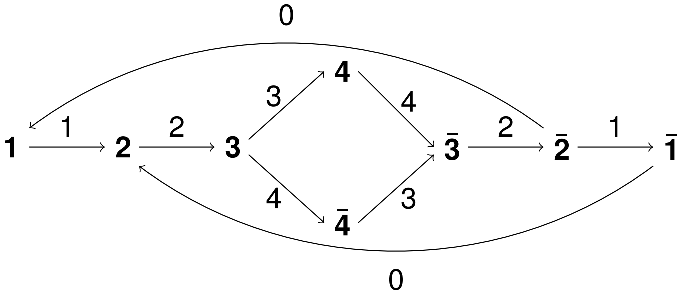
Example: level 1 perfect crystal for $D_4^{(1)}$



Affine Crystals

Theorem ([KKMMNN 92])

Fix a positive integer $l > 0$ and let $\mathcal{B}$ be a perfect crystal of level l. For any classical dominant weight $\lambda \in \bar{P}_l^+$, there exists a unique crystal isomorphism

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}$$

given by $u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$, where b_λ is the unique element in $\mathcal{B}$ such that $\varphi(b_\lambda) = \lambda$.

Path Model (cont.)

Set $\lambda_0 = \lambda, \lambda_{k+1} = \varepsilon(\lambda_k), b_0 = b_\lambda, b_{k+1} = b_{\lambda_{k+1}}$. The sequences

$$\mathbf{w}_\lambda := (\lambda_k)_{k=0}^{\infty}, \mathbf{b}_\lambda := (b_k)_{k=0}^{\infty},$$

are periodic with the same period N.

Definition

1. The sequence $\mathbf{b}_\lambda$ is called the *ground-state path* of weight λ.
2. A λ-*path* in $\mathcal{B}$ is a sequence $\mathbf{p} = (p_k)_{k=0}^{\infty}$ with $p_k = b_k$ for all $k \gg 0$.

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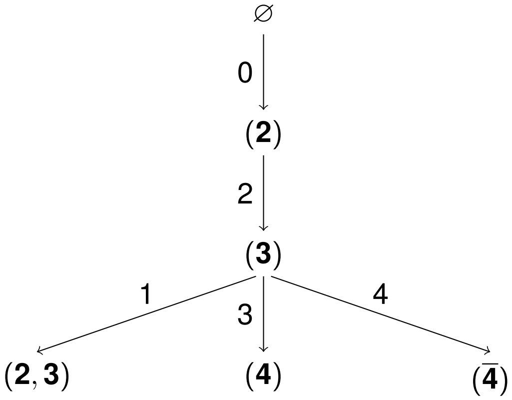
Example: top part of $\mathcal{B}(\Lambda_0)$ for $D_4^{(1)}$



We now consider only $\mathfrak{g} = HD_n^{(1)}$.

- The *degree* of a root $\alpha = -\sum_{i=-1}^{n} a_i \alpha_i$ is defined to be a_{-1} and is denoted $\deg(\alpha)$.
- Consider the element $-l\alpha_{-1} - k\delta \in Q^-$, where $\delta = \alpha_0 + \alpha_1 + 2\sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n$. Then we have the following:

Theorem ([Klima Misra 08] for $C_n^{(1)}$, analogous proof)

$\alpha = -l\alpha_{-1} - k\delta$ is a root of $HD_n^{(1)}$ only if $k \geq l$. If $l = k$ then $mult(\alpha) = n$.

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$\alpha = -l\alpha_{-1} - k\delta$ is a root of $HD_n^{(1)}$ only if $k \geq l$. If $l = k$ then $mult(\alpha) = n$.

A Lemma on $W(S)$.

Here are the elements of $W(S)$ of length 1 and 2:

$\ell(\omega)$	ω	$\omega\rho - \rho$	level
1	r_{-1}	$-\alpha_{-1} = \Lambda_0$	1
2	$r_{-1}r_0$	$-2\alpha_{-1} - \alpha_0 = \Lambda_2 - \delta$	2

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Roots of degree 1

If $\alpha = -\alpha_{-1} - k\delta$ then $\mathrm{mult}(\alpha) = \dim(V(\Lambda_0)_\alpha)$. These are given for $D_n^{(1)}$ by the following generating series:

$$\sum_{k=0}^{\infty} \dim(V(\Lambda_0)_{\Lambda_0 - k\delta}) q^k = \prod_{i=1}^{\infty} (1 - q^i)^{-n}.$$

Using the binomial expansion $(1 - q^i)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j} q^{ij}$, we can in principle compute the multiplicity of α for any k. In particular, we see that it is a polynomial in n of degree k.

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The first few multiplicity polynomials are given in the following table:

Root	Multiplicity
$-\alpha_{-1} - \delta$	n
$-\alpha_{-1} - 2\delta$	$\frac{n(n+3)}{2}$
$-\alpha_{-1} - 3\delta$	$\frac{n(n+1)(n+8)}{6}$
$-\alpha_{-1} - 4\delta$	$\frac{n(n+1)(n+3)(n+14)}{24}$
$-\alpha_{-1} - 5\delta$	$\frac{n(n+3)(n+6)(n^2+21n+8)}{120}$

Degree 2 Roots

For $\tau \in P_2^+$ let

$$X(\tau) = \sum_{\lambda < \tau} \dim(V(\Lambda_0)_\lambda) \dim(V(\Lambda_0)_{\tau-\lambda}).$$

Then Kang's formula gives:

$$\mathrm{mult}(-2\alpha_{-1} - 3\delta) = X(2\Lambda_0 - 3\delta) - \dim(V(\Lambda_2 - \delta)_{2\Lambda_0 - 3\delta}).$$

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Consider the root $\alpha = -2\alpha_{-1} - 3\delta$.

$$\text{Let } \lambda_i = \begin{cases} \Lambda_i - \Lambda_{i-1} \text{ if } 1 \leq i \leq n, i \neq 2, n-1, \\ \Lambda_2 - \Lambda_0 - \Lambda_1 \text{ if } i = 2, \\ \Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n \text{ if } i = n-1. \end{cases}$$

λ	$\dim(V(\Lambda_0)_\lambda)$	$\dim(V(\Lambda_0)_{\alpha-\lambda})$	Count
Λ_0	1	$\frac{n(n+1)(n+8)}{6}$	1
$\Lambda_0 \pm \lambda_i \pm \lambda_j - \delta, i < j$	1	n	$\frac{4n(n-1)}{2}$
$\Lambda_0 - \delta$	n	$\frac{n(n+3)}{2}$	1

We have: $X(2\Lambda_0 - 3\delta) = \frac{n(n+1)(n+8)}{6} + \frac{n^2(5n-1)}{2}$.

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Theorem (Misra, W., 2012)

$$dim(V(\Lambda_2)_{2\Lambda_0-2\delta}) = \frac{n^2(5n-1)}{2}$$

Therefore,

$$\begin{aligned} \text{mult}(-2\alpha_{-1}-3\delta) &= X(2\Lambda_0-3\delta) - \dim(V(\Lambda_2-\delta)_{2\Lambda_0-3\delta}), \\ &= \frac{n(n+1)(n+8)}{6} + \frac{n^2(5n-1)}{2} - \frac{n^2(5n-1)}{2}, \\ &= \frac{n(n+1)(n+8)}{6}. \end{aligned}$$

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Conjecture (Misra, W.)

Let α be a root of $HD_4^{(1)}$ of degree 2. Then mult(α) depends only on $1 - \frac{(\alpha|\alpha)}{2}$

If true, then the following is a generating series for the root multiplicities:

$$\sum_{k=0}^{\infty} f(k)q^k = \left(\sum_{k=0}^{\infty} p^4(k)q^k\right)(1-3q^8+7q^{10}-15q^{12}+30q^{14}-\cdots)$$

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Possible future work

- Further areas to explore:
  - Proving the conjecture of polynomial behavior of root multiplicities for $HD_n^{(1)}$ of fixed degree k.
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