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THE COVARIANT PHASE SPACE OF ASYMPTOTICALLY FLAT GRAVITATIONAL FIELDS

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Lagrange appears to be the first to have recognized that phase spaces of physical systems can be constructed from entire *histories*, without reference to a particular instant of time. We begin with a brief review of these covariant constructions for field theories. We point out that, in the case of general relativity, boundary conditions play a critical role and must be adjusted carefully for the symplectic structure to be finite, and hence for the framework to be well-defined. We then present a new application: the derivation of the expression of energy-momentum of an isolated gravitating system at *null* infinity. This derivation makes a crucial use of the covariant construction and cannot be carried out within the familiar, $3+1$ phase space frameworks.

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1 INTRODUCTION

The phase space description of classical systems has proved to be a powerful tool in the analysis of their physical properties. The key reason is that phase spaces naturally arise as symplectic manifolds. The availability of the symplectic structure enormously simplifies the task of specifying the dynamical flow: in place of a vector field, one now has to specify only its generating function, the Hamiltonian. There are simplifications also in the reverse direction. For, it is often the case that while one can easily identify symmetries of physical systems, it is harder to obtain expressions of the corresponding conserved quantities. In the phase space framework, these can be obtained as the generators of canonical transformations corresponding to those symmetries. (An example of this situation is discussed in Section 5 of this article.) This two-way procedure is indeed one of the most elegant and powerful applications of geometry to physics.

However, phase spaces are generally constructed by decomposing space-time into space and time, and Hamiltonian mechanics is therefore widely perceived as being non-covariant. In field theories in Minkowski space-time, for example, considerable effort is devoted to showing that, in spite of the non-covariant nature of the phase space, the Poincaré group does act on it in a meaningful way. This complicated procedure is completely unnecessary. For, it has been known for quite some time now that one can construct phase spaces from entire *histories* without ever having to introduce preferred instants of time. The idea is old indeed. In fact, it can be traced[1] back all the way to Lagrange![2] It therefore seems most appropriate to discuss the covariant phase space framework in a volume dedicated to him.† The purpose of this article is to review this framework and to illustrate its power by means of an example, which, to our knowledge, has not been discussed in the literature.

† In recent years, covariant phase spaces have appeared in a number of books. Examples are: Souriau's text on dynamics,[1] Chernoff and Marsden's lecture notes on infinite dimensional Hamiltonian systems,[3] Schutz's monograph on geometry and physics,[4] and Woodhouse's on geometric quantization.[5] Quantization of linear fields in curved space-times based on such a covariant Hamiltonian formulation is discussed, *e.g.*, in Ref. 6. The framework has also been used in general relativity in the stability analysis of stars and black holes,[7,8] for the construction of the phase space of radiative modes of the gravitational field in the full, non-linear theory,[9,10] in the definition of conserved quantities,[11–13] and in the analysis of the group of local symmetries.[14]

Let us begin with an overview of the situation.

In terms of the familiar, non-covariant constructions, the basic ideas of the covariant procedure can be summarized as follows. When the initial value problem is well-posed, each dynamical trajectory in the configuration space is completely specified by its initial data, which in turn correspond just to a point in the phase space. Hence, by fixing an instant of time, one obtains an isomorphism between the space of solutions to the dynamical equations —*i.e.*, of dynamically allowed histories— and the phase space of the system. One can therefore pull-back the symplectic structure to the space of solutions. Since the symplectic 2-form on the phase space is Lie-dragged by the dynamical vector field, the symplectic structure we thus obtain on the space of solutions is independent of the choice of the initial instant of time. Thus, the space of solutions is in fact equipped with a *natural* symplectic structure —called by Souriau[1] the *Lagrange form* to emphasize the fact that Lagrange was the first to recognize its existence. This is the covariant phase space of the system. It captures all the relevant classical physics without the necessity of selecting a preferred instant of time. Typically, however, this space does *not* have a natural cotangent bundle structure. It is only when one is given an instant of time that one can identify each of its points with an initial data set and represent it as a cotangent bundle. However, Hamiltonian mechanics does not *need* a cotangent bundle; it suffices to have just a symplectic manifold for its arena. And this is precisely the structure that naturally exists on the covariant phase space.

How can one speak of dynamics in this setting? After all, a point of the covariant phase space is an entire history. How can things "evolve" then? The answer is that the familiar evolution appears in the disguise of a mapping between histories. Consider, for example, a Klein-Gordon field in Minkowski space. The time translation subgroups of the Poincaré group act on the space of solutions to this equation and this action can be interpreted as "time evolution." Under this motion, the symplectic structure is left invariant and the generator of this canonical transformation is precisely the familiar energy function. More generally, given a notion of time, one can introduce the notion of dynamics on the covariant phase space as follows. Look for initial data of all dynamical trajectories and map each trajectory to another one, whose initial data at time t_0 are the same as the initial data of the first trajectory at time t_1. In the covariant framework, this mapping between *entire histories* can be interpreted as the "time-evolution" from t_0 to t_1. Its generating function turns out to be usual Hamiltonian.

In gauge theories, the non-covariant Hamiltonian description has first class constraints; not every point of the phase space represents permissible initial data. Furthermore, even if one is given initial data satisfying these constraints, one does not obtain a unique dynamical trajectory in the phase space because one can perform arbitrary gauge transformations in the course of evolution. More precisely, the first class constraints generate gauge transformations, and the Hamiltonian is unique only up to the addition of constraint functionals. How are these features incorporated in the covariant description? If one repeats the construction given above to pass to the covariant picture from the non-covariant one, one finds that the space of solutions is now naturally equipped with a degenerate symplectic structure — or, in the standard terminology, a *presymplectic structure*. Motions along the degenerate directions are precisely the gauge transformations of the theory. Note that, since each point of the covariant phase space is a solution to *all* field equations, there is no longer a constraint surface in the phase space nor a constraint functional to generate gauge. The gauge directions are coded directly in the degeneracy of the presymplectic structure. Finally, the arbitrariness of the dynamical evolution also results from this degeneracy; the Hamiltonian itself is unambiguous.

In Section 2, we recall the general framework for field theories on a background space-time. Since the covariant approach deals with histories rather than Cauchy data, it is now easier to see the relation between the Lagrangian and the Hamiltonian descriptions. In Section 3, we introduce the covariant Hamiltonian description of gravitational fields (in general relativity) which are asymptotically flat at spatial infinity. In the standard, $3+1$ phase space formulation, while it is easy to see the role of spatial diffeomorphisms, the role of space-*time* diffeomorphisms is rather obscure. Indeed, as is well-known,[15] because the Poisson algebra of constraints is open in the BRST sense, it is considerably larger than the Lie algebra of space-time diffeomorphisms. In the covariant approach, on the other hand, the issue of diffeomorphisms is rather straightforward. Because we avoid a $3+1$ decomposition, we can treat *all* space-time diffeomorphisms on the same footing. However, to fully analyse this issue, we have to specify the boundary conditions carefully. In fact, already the integrals defining the symplectic structure itself diverge unless the boundary conditions are delicately adjusted. Since this issue is overlooked in the earlier covariant treatments, we will spell out the boundary conditions and point out the precise role they play. In section 4, we show that the ADM 4-momentum[16] is the generator of the asymptotic translation group which arises from the boundary conditions. Once again, the result differs from those obtained in the well known $3+1$ treatments. There, the Hamiltonians generating space-time translations are expressible as a sum of a volume term —suitably smeared constraints— and a surface term, the ADM 4-momentum integrals. In the covariant treatment, the surface integrals alone generate asymptotic translations. This final result is not new. The novelty of our treatment lies only in the careful specification of boundary conditions and avoidance of divergent integrals in the intermediate steps.

In Section 5, on the other hand, we analyse a new situation: space-times which are asymptotically flat at *null* infinity. It turns out that the introduction of the symplectic structure and the analysis of its properties in this case is completely analogous to that discussed in Section 4. The only difference is that now the Bondi-Metzner-Sachs (BMS)[17] group replaces the Poincaré group as the group of asymptotic symmetries. Therefore, instead of repeating the previous analysis, we shall focus only on the differences. The final result is the following: we show that the generator of asymptotic BMS translations can also be expressed as a sum of a 2-sphere integral — the super-momentum evaluated at a cross-section of null infinity— and a volume integral —the total super-momentum carried by gravitational waves until the retarded instant of time represented by the cross-section. Using the result of Section 4, we can therefore conclude that in space-times which are asymptotically flat both at null and spatial infinity, the ADM 4-momentum equals the sum of the Bondi 4-momentum[17] at a retarded instant of time and the total 4-momentum carried away by the gravitational waves until that instant. (An independent proof was given in Ref. 18). Thus, apart from clarifying certain issues, the covariant approach leads one to results which simply cannot be obtained in the $3+1$ treatments; it enables one to define quantities at null infinity and relate them to those at spatial infinity.

We will use units in which only c is set equal to 1. Our other conventions are as follows. Penrose's abstract index notation is used for tensor fields on space-time, while index-free notation is used for fields on infinite dimensional spaces, such as the phase space. The space-time signature is $(-,+,+,+)$. The Riemann tensor is defined by $\nabla_{[a}\nabla_{b]}k_c \equiv \frac{1}{2}(\nabla_a\nabla_b - \nabla_b\nabla_a)k_c =: \frac{1}{2}R_{abc}{}^d k_d$, and the Ricci tensor by $R_{ab} := R_{amb}{}^m$. The emphasis will be on presenting the overall conceptual structure; the details of calculations will be omitted. Also, the treatment will not be rigorous with respect to functional analysis.

2 PRELIMINARIES

Fix a smooth 4-dimensional manifold M, which is topologically $\Sigma \times \mathbb{R}$, where (the complement of a compact set in) Σ is diffeomorphic to (the complement of a compact set in) $\mathbb{R}^3$. We will assume that M is equipped with a non-dynamical, globally hyperbolic metric $\mathring{g}_{ab}$ whose Cauchy surfaces are diffeomorphic to Σ. On this space-time, consider a dynamical theory for a collection of fields $\phi^\alpha(x)$, where α is a collective label for the fields, which may include space-time indices and/or internal indices. Denote by $\mathcal{F}$ the infinite-dimensional manifold of smooth fields ϕ^α on M satisfying suitable boundary conditions. (These will be specified only in the detailed discussion of general relativity to be considered in the remaining sections.) Functions on $\mathcal{F}$ are functionals $f[\phi]$.

Our basic assumption will be that dynamics is specified by some action. Thus, given any measurable region $V \subset M$ and any collection of fields $\phi \in \mathcal{F}$ we have an action integral,

$$S_V[\phi] = \int_V L(\phi, \mathring{\nabla}\phi, \ldots)\, \mathrm{d}^4\mathring{v}, \tag{2.1}$$

such that second order field equations for ϕ^α are obtained by requiring that S_V be stationary under any variation X^α of ϕ^α which vanishes on the boundary ∂V. We shall allow the Lagrangian L to contain terms which are pure divergences; the action may thus have some surface terms. The variation of S_V can be expressed as the gradient of S_V in $\mathcal{F}$ applied to X^α, which, in general, is of the form

$$\mathrm{d}S_V(X) = \int_V G_\alpha(\phi) X^\alpha\, \mathrm{d}^4\mathring{v} + \oint_{\partial V} F^a(\phi, X)\, \mathrm{d}\mathring{S}_a, \tag{2.2}$$

where G_α, whose vanishing gives the field equations, depends on derivatives up to second order of ϕ^α, and F^a vanishes when $X^\alpha = 0$ on ∂V. (Note that, in this procedure, we are regarding S_V as a differentiable function on $\mathcal{F}$ even when the surface term in (2.2) is non-zero. That is, we are allowing the gradient of S_M to be a distribution on V.) *Our phase space will now be the submanifold* Γ *of* $\mathcal{F}$ *consisting of fields* ϕ *which extremize* S_M, i.e. *which satisfy the field equations* $G_\alpha = 0$. We indicate the natural embedding of this submanifold by $i : \Gamma \to \mathcal{F}$.

Given a Cauchy surface Σ, consider the 1-form θ_Σ on Γ defined by

$$\theta_\Sigma(X) := \int_\Sigma F^a(\phi, X)\, \mathrm{d}\mathring{S}_a, \tag{2.3}$$

for all vector fields X. This form will serve as a *potential* for our presymplectic structure. In general, θ_Σ *does* depend on the choice of Σ (although, as we shall see below, one can often choose a potential which is invariant under certain changes of hypersurfaces). Let us now consider the case when V is bounded by two Cauchy surfaces Σ and Σ' connected by a timelike world tube $\mathcal{T}_\infty$ at spatial infinity. Then, using (2.2) and (2.3), we have:

$$i^*\mathrm{d}S_V = \theta_{\Sigma'} - \theta_\Sigma + \theta_{\mathcal{T}_\infty}. \tag{2.4}$$

Hence, the exterior derivative ω_Σ of θ_Σ,

$$\omega_\Sigma(X, Y) := \mathrm{d}\theta_\Sigma(X, Y), \tag{2.5}$$

has a rather simple behavior under the change of the Cauchy surface:

$$0 \equiv i^*\mathrm{d}^2 S_V = \omega_{\Sigma'} - \omega_\Sigma + \omega_{\mathcal{T}_\infty}. \tag{2.6}$$

Thus, the current whose integral over Σ gives the 2-form ω_Σ on Γ is conserved due to the equations of motion satisfied by the fields ϕ and their perturbations X, Y. Therefore, *provided* our boundary conditions are such that they force the term $\omega_{\mathcal{T}_\infty}$ to vanish, we would find that, unlike its potential θ_Σ, the 2-form ω on Γ is in fact *independent* of the choice of the Cauchy surface. Furthermore, being the exterior derivative of the 1-form θ_Σ, the 2-form ω is closed. *This is the presymplectic structure —or, the Lagrange form— on the space of solutions* Γ *of the field equations that we are seeking.* We will see that, in the gravitational case to be treated in detail, the boundary conditions do indeed satisfy the required condition. Finally, note that we could have defined the one-form θ_Σ and its exterior derivative ω_Σ on all of $\mathcal{F}$. However, in the absence of field equations, we would have found that the current defining the 2-form fails to be conserved, whence the 2-form has a dependence on the Cauchy surface which can not be eliminated just by a suitable choice of boundary conditions.

In general, the presymplectic 2-form so defined on Γ *is* degenerate and, as pointed out in the introduction, motions along these degenerate directions correspond to gauge transformations of the theory. Denote by $N \subset T\Gamma$ the set of degenerate directions X: $\omega(X,Y) = 0$ for all $Y \in T\Gamma$. Then the Lie derivative of ω along such directions is always zero: $X \in N$ implies

$$\mathcal{L}_X\omega \equiv \mathrm{d}(\mathrm{i}_X\omega) + \mathrm{i}_X\mathrm{d}\omega = 0. \tag{2.7}$$

Furthermore, the degenerate flats N are integrable: if X and $\bar{X}$ are two vector fields on Γ which lie in the degenerate flats everywhere, and Y is an arbitrary vector field, using (2.7) we have:

$$\omega(\mathcal{L}_X\bar{X}, Y) = \mathcal{L}_X(\omega(\bar{X},Y)) - \omega(\bar{X}, \mathcal{L}_XY) = 0, \tag{2.8}$$

where in the last step we have used the fact that $\bar{X}$ is is a degenerate direction of ω. Therefore, one can quotient Γ by the integral manifolds of the degenerate flats N and obtain a natural (non-degenerate) symplectic structure on the resulting reduced phase space.

Remarks:

1) In the standard situation, when the action is of first order, $S = \int_V L(\phi, \mathring{\nabla}\phi)\,\mathrm{d}^4\mathring{v}$, one can obtain[5,14] an explicit expression for the forms θ_Σ and ω in terms of the Lagrangian L. Now, the variation of the action is given by

$$\mathrm{d}S(X) = \int_V \left(\frac{\partial L}{\partial\phi^\alpha} - \mathring{\nabla}_a\frac{\partial L}{\partial\phi^\alpha_a}\right) X^\alpha\,\mathrm{d}^4\mathring{v} + \oint_{\partial V}\frac{\partial L}{\partial\phi^\alpha_a}X^\alpha\,\mathrm{d}\mathring{S}_a, \tag{2.9}$$

where $\phi^\alpha_a := \mathring{\nabla}_a\phi^\alpha$, whence the presymplectic potential 1-form now becomes $(\theta_\Sigma)_\phi(X) = \int_\Sigma(\partial L/\partial\phi^\alpha_a)X^\alpha\,\mathrm{d}\mathring{S}_a$, resembling in form the expression $\theta = p\,\mathrm{d}q$ from mechanics. Taking the derivative, we obtain the symplectic structure:

$$\begin{aligned}(\omega)_\phi(X,Y) := (\mathrm{d}\theta_\Sigma)_\phi(X,Y) = \int_\Sigma\Big[&\frac{\partial^2 L}{\partial\phi^\beta\partial\phi^\alpha_a}(Y^\alpha X^\beta - X^\alpha Y^\beta)\\ &+ \frac{\partial^2 L}{\partial\phi^\beta_b\partial\phi^\alpha_a}(Y^\alpha\mathring{\nabla}_b X^\beta - X^\alpha\mathring{\nabla}_b Y^\beta)\Big]\mathrm{d}\mathring{S}_a.\end{aligned} \tag{2.10}$$

These expressions have been useful in a number of applications.[3–8]

2) In classical mechanics, it is only the (pre)-symplectic 2-form that is relevant; in general its potentials do not have a direct physical significance. However, sometimes the use of potentials does simplify calculations. A particularly useful example is the calculation of Hamiltonians generating a given infinitesimal canonical transformation X. By definition of an infinitesimal canonical transformation, the Lie derivative of ω by X vanishes. Of course, the same is not true of a general symplectic potential. If, on the other hand, we manage to find a symplectic potential θ which *is* Lie-dragged by X, the identity $\mathcal{L}_X\theta \equiv \mathrm{i}_X\omega + \mathrm{d}(\theta(X)) = 0$ implies that (modulo an additive constant) $H = \theta(X)$ is the unique Hamiltonian generating the given canonical transformation.

3) In field theory, one is often interested in the canonical transformations associated with space-time diffeomorphisms. Suppose that ξ^a is a vector field on space-time such that $\mathcal{L}_\xi\phi^\alpha$ is a solution of the linearized field equations around ϕ^α, satisfying the boundary conditions defining $\mathcal{F}$ (or Γ). Then $X_{(\xi)} := \mathcal{L}_\xi\phi$ can be considered as a vector field on Γ, generating a 1-parameter family of diffeomorphisms $\sigma_t : \Gamma \to \Gamma$. Let us examine the behavior of a symplectic potential θ_Σ under the action of these diffeomorphisms. By definition of Lie derivative, we have:

$$\begin{aligned}(\mathcal{L}_{X_{(\xi)}}\theta_\Sigma)_\phi(Y) :&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\sigma^*_{-\epsilon}\theta_\Sigma)_\phi(Y) - (\theta_\Sigma)_\phi(Y)\right]\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\theta_\Sigma)_{\sigma_\epsilon\phi}(\sigma_{\epsilon *}Y) - (\theta_\Sigma)_\phi(Y)\right].\end{aligned} \tag{2.11}$$

Now, if the background non-dynamical metric $\mathring{g}_{ab}$ is Lie-dragged by ξ^a, then this last expression can be written in terms of the current F^a (of (2.3)) as

$$\begin{aligned}(\mathcal{L}_{X_{(\xi)}}\theta_\Sigma)_\phi(Y) &= \int_\Sigma(\mathcal{L}_\xi F^a)\,\mathrm{d}\mathring{S}_a\\ &= \oint_{\partial\Sigma}F^a\xi^b\,\mathrm{d}\mathring{S}_{ab} + \int_\Sigma(\mathring{\nabla}_b F^b)\,\xi^a\mathrm{d}\mathring{S}_a.\end{aligned} \tag{2.12}$$

Therefore, in these cases, the Lie-derivative of θ_Σ by $X_{(\xi)}$ vanishes if the surface integral in (2.12) is zero *and* ξ^a is tangent to Σ or the current F^a is conserved. If θ_Σ does have this property, then, by remark 2) above, the Hamiltonian generating the infinitesimal canonical transformation $X_{(\xi)}$ on the phase space is given by $\theta(X_{(\xi)})$. Consider, as an example, a Klein-Gordon field

ϕ in Minkowski space-time. In this case, for reasonable boundary conditions on ϕ, the potential 1-form defined by $\theta_\Sigma(Y) := -\frac{1}{2}\int_\Sigma [(\nabla_a\phi)\, Y - \phi\,(\nabla_a Y)]\, \mathrm{d}S^a$ satisfies all these properties for all Killing fields ξ^a of the Minkowski metric; it is Poincaré invariant. This enables one to write out the expressions of the generators of the Poincaré group simply as: $-\frac{1}{2}\int_\Sigma [(\nabla_a\phi)(\mathcal{L}_\xi\phi) - \phi\,(\nabla_a \mathcal{L}_\xi\phi)]\,\mathrm{d}S^a$.

4) We can now use the previous remark to go back to the issue of the dependence of θ_Σ on the hypersurface Σ. From (2.11) we also obtain that

$$\begin{aligned}(\mathcal{L}_{X_{(\xi)}}\theta_\Sigma)_\phi(Y) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\theta_{\sigma_\epsilon(\sigma_{-\epsilon}\Sigma)})_{\sigma_\epsilon\phi}(\sigma_{\epsilon}{*}Y) - (\theta_\Sigma)_\phi(Y)\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\theta_{\sigma_{-\epsilon}\Sigma})_\phi(Y) - (\theta_\Sigma)_\phi(Y)\right],\end{aligned} \tag{2.13}$$

where we have again assumed, in the last step, that the background field $\mathring{g}_{ab}$, if it appears explicitly in the integrand giving θ_Σ, is Lie-dragged by ξ^a. We see therefore that an alternative condition for the Lie-derivative of a potential θ_Σ by $X_{(\xi)}$ to vanish is that θ_Σ be left unchanged under the replacement of Σ by its image under the space-time diffeomorphism generated by ξ^a. Thus, the potential in our example above for a Klein-Gordon field in Minkowski space-time is Poincaré invariant also in the sense of the action of the Poincaré group on Σ.

5) Finally, we note that the steps leading to (2.13) above can also be repeated for a symplectic structure. Since ω is manifestly independent of Σ, we conclude that if the background fields that possibly appear in its expression are Lie-dragged by ξ^a, ω is necessarily Lie-dragged by $X_{(\xi)}$; the motions generated by $X_{(\xi)}$ on the phase space are necessarily canonical transformations.

3 THE COVARIANT PHASE SPACE OF GENERAL RELATIVITY

We now turn to general relativity. In this section we will focus on gravitational fields which are asymptotically flat at spatial infinity. We begin in the subsection 3.1 by stating the precise fall-off conditions that we will assume. In 3.2, we derive the expression of the presymplectic structure following the procedure outlined in section 2. In 3.3, we analyse some of its basic properties which, in section 4, will enable us to express the ADM 4-momentum as the generator of space-time translations.

3.1 Boundary conditions

Fix, as before, a 4-manifold M which is topologically $\Sigma \times \mathbb{R}$, where (the complement of a compact subset of) the 3-manifold Σ is diffeomorphic to (the complement of a compact set in) $\mathbb{R}^3$. The set $\mathcal{F}$ of histories of interest is now to consist of smooth, globally hyperbolic, Lorentzian metrics g_{ab} on M whose Cauchy surfaces are diffeomorphic to Σ and which are asymptotically flat at spatial infinity. We shall see in the next subsection that unless the boundary conditions are carefully adjusted, the integral defining the presymplectic form does not converge and hence the Hamiltonian description fails to be meaningful. Therefore, in this subsection, we make a small detour to specify the precise fall-off we will assume on the space-time metric g_{ab}.

Since our aim is to construct a covariant Hamiltonian framework, we must use a 4-dimensional notion of asymptotic flatness. Thus, our task is to specify how the space-time metric is to behave as we approach infinity in *any* spacelike direction, not necessarily restricted to lie within a given Cauchy surface. This can be done in a manifestly coordinate-invariant fashion by attaching to space-time a certain boundary and specifying the behavior of the metric at this boundary.[19] However, since this is a recent construction, and since the techniques it uses may be unfamiliar to readers interested in symplectic geometry, we will present an essentially equivalent set of boundary conditions —involving asymptotic coordinate systems— which is easier to grasp intuitively.

Let us begin by recalling the situation in Minkowski space. Here, one can approach spatial infinity along spacelike geodesics. To take limits of physical fields as one goes to infinity following these geodesics, it is convenient to introduce a set of coordinates $(\rho, \tau, \theta, \phi)$ where θ and ϕ are the standard spherical coordinates and ρ and τ are related to the standard r and t via: $t = \rho \sinh\tau$ and $r = \rho\cosh\tau$. Then, outside the light cone of the origin, Minkowski space is foliated by a 1-parameter family of timelike hyperboloids, $\rho = const$, and spatial infinity is represented by the limiting hyperboloid $\mathcal{H}$ defined by "$\rho = \infty$." $\mathcal{H}$ can be reached by going to infinity along radial spacelike geodesics, curves on which (τ, θ, ϕ) remain constant. Outside the light cone of the origin, the Minkowskian metric η_{ab} can be expressed as:

$$\eta_{ab} = \rho_a\rho_b + \rho^2 h^0_{ab}, \tag{3.1}$$

where h^0_{ab} is the induced metric on the unit timelike hyperboloid $\rho = 1$ and $\rho_a := \partial_a \rho$. The idea now is to impose asymptotic flatness at spatial infinity by requiring that the components of the space-time metric g_{ab} in this chart approach those of η_{ab} at spatial infinity at an appropriate rate. There exists in the literature an extensive discussion on the issue of what the appropriate rate is in this 4-dimensional setting (see *e.g.* Refs. 20, 21). Here, we will only state the resulting boundary conditions in the coordinate language introduced above[21] and briefly discuss their physical implications.

The metric g_{ab} will be said to be asymptotically flat at spatial infinity if it admits an expansion of the type:

$$g_{ab} = \Big(1 + \frac{\sigma}{\rho}\Big)^2 \rho_a \rho_b + \rho^2 \Big(h^0_{ab} - 2\sigma \frac{h^0_{ab}}{\rho}\Big) + \mathcal{R}_{ab}, \tag{3.2}$$

where σ is a function on the (unit) hyperboloid $\mathcal{H}$, *even* under the natural inversion mapping (induced by the Minkowskian transformation $x^\mu \to -x^\mu$), and where, in the Cartesian coordinates associated with our chart, the components of the "remainder," $\mathcal{R}_{ab}$, fall off at least as fast as $1/\rho^2$. The content of these conditions can be explained as follows. First of all, we are requiring that g_{ab} approach η_{ab} as $1/\rho$. Secondly, the first-order deviation from η_{ab} is required to satisfy three conditions: (i) there is no mixed "radial-angular" component to this order; (ii) the ρ-ρ component —the mass aspect— has *even* parity under reflection; and, (iii) the hyperboloid component is completely determined by the ρ-ρ component and the unit hyperboloid metric; it is precisely $-2\sigma h^0_{ab}$. The first condition is indeed very mild; if g_{ab} does not satisfy it, one can always find a diffeomorphically related metric which will satisfy this condition. The last two conditions ensure that the asymptotic symmetry group is the Poincaré group. The second condition rules out the possibility of making what is known as "logarithmic translations" while the third rules out the so-called "spi supertranslations." (For details, see Refs. 20, 21). These restrictions are not stringent in the sense that there is a large class of solutions to Einstein's equations which satisfies them. Roughly speaking, any solution which one intuitively thinks of as being asymptotically flat at spatial infinity can be made to satisfy the definition by exploiting the diffeomorphism freedom. In essence, the failure to thus satisfy (iii) would occur only if the solution is of NUT-type, carrying a non-zero "magnetic mass," and hence fails to be asymptotically flat in the intuitive sense. We know of no solution in which (ii) fails. It is quite likely that the covariant phase space analysis can go through under even weaker assumptions. However there is no obvious physical ground to weaken the assumptions.

3.2 *The presymplectic 2-form*

Following the procedure outlined in section 2, we can now define the covariant phase space of general relativity to be the subspace Γ of $\mathcal{F}$ consisting of solutions g_{ab} of Einstein's (vacuum) equations. Thus, in particular, each element of Γ is asymptotically flat. Our task now is to introduce on Γ the appropriate presymplectic structure.

The first step in this procedure is the specification of the action. It is convenient to begin with the first order, Palatini formalism in which the metric g_{ab} and the connection ∇ are at first independent and the action is given by:

$$\begin{aligned} S_V[g,\nabla] &= \frac{1}{2k}\int_V \mathrm{d}^4x\,\sqrt{-g}\,g^{ab}R_{ab}(\nabla) \\ &\equiv \frac{1}{k}\int_V \mathrm{d}^4x\,\sqrt{-g}\,g^{ab}(\partial_{[m}\Gamma^m{}_{a]b} + \Gamma^n{}_{b[a}\Gamma^m{}_{m]n}), \end{aligned} \tag{3.3}$$

where we have set $k = 8\pi G$. Let us compute the potential θ_Σ for the presymplectic structure. A tangent vector γ to $\mathcal{F}$ is now represented by a pair $\gamma = (\delta g, \delta\Gamma)$ of fields on M satisfying the appropriate boundary conditions. Following the procedure outlined in section 2 (see especially equations (2.9)–(2.10)), one can compute the action of θ_Σ on a general tangent vector γ at a point g_{ab} of $\mathcal{F}$. The result is:†

$$(\theta_\Sigma)_{(g,\Gamma)}(\gamma) = \frac{1}{k}\int_\Sigma g^{m[n}\delta\Gamma^{p]}{}_{mn}\,\mathrm{d}S_p. \tag{3.4}$$

Taking its exterior derivative, one obtains the presymplectic structure ω. Its

† With our boundary conditions, the integral defining this presymplectic potential need not converge. Finiteness could have been guaranteed by adding to the action a suitable boundary term. Such additions do not affect the presymplectic structure itself, which, as we shall see *is* finite. Therefore, for the simplicity of presentation we have refrained from adding boundary terms.

action on two tangent vectors γ and γ' at the point $(g_{ab}, \Gamma^a{}_{bc})$ is given by:

$$(\omega)_{(g,\Gamma)}(\gamma,\gamma') = \frac{1}{2k}\int_\Sigma \mathrm{d}S_c\Big[2\,\delta'\Gamma^{[c}{}_{ab}\,(\delta g^{a]b} + \tfrac{1}{2}\,g^{a]b}\delta\ln,g,) \\ - 2\,\delta\Gamma^{[c}{}_{ab}\,(\delta' g^{a]b} + \tfrac{1}{2}\,g^{a]b}\delta'\ln,g,)\Big]. \tag{3.5}$$

Recall, however, that we are only interested in the pull-back of this 2-form on $\mathcal{F}$ to the phase space Γ. Since for points in Γ the covariant derivative operator ∇ is compatible with the metric g_{ab}, the $\delta\Gamma^c{}_{ab}$ term in γ is completely determined by δg_{ab}:

$$\delta\Gamma^c{}_{ab} = \tfrac{1}{2}\,g^{cd}(\nabla_a\,\delta g_{db} + \nabla_b\,\delta g_{ad} - \nabla_d\,\delta g_{ab}). \tag{3.6}$$

Hence, from now on, we will denote the tangent vectors γ to the phase space just by $\gamma_{ab} \equiv \delta g_{ab}$. Substituting (3.6) into (3.5), we obtain, for all g_{ab} in Γ and tangent vectors γ and γ' to Γ,

$$(\omega)_g(\gamma,\gamma') = \frac{1}{4k}\int_\Sigma\Big[\epsilon^{anc}{}_q\,\epsilon^{bpdq}(\gamma_{ab}\nabla_n\gamma'_{cd} - \gamma'_{ab}\nabla_n\gamma_{cd}) + 2\nabla_a(\gamma^{b[p}\gamma'^{a]}{}_b)\Big]\mathrm{d}S_p, \tag{3.7}$$

where ϵ_{abcd} is the volume form associated with g_{ab}. Finally, we use the boundary conditions satisfied by various fields to further simplify this expression. Since each g_{ab} in Γ has, in particular, the fall-off given by (3.2), the tangent vectors γ_{ab} have the asymptotic form:

$$\gamma_{ab} = \frac{2\,\delta\sigma}{\rho}\,(\rho_a\rho_b - \rho^2 h^0_{ab}) + \mathcal{R}_{ab}, \tag{3.8}$$

where, as before, the components of the remainder $\mathcal{R}_{ab}$ in the asymptotic Cartesian chart fall off as $1/\rho^2$. With this information at hand, let us return to (3.7). Since the last term on its right hand side is a total divergence, it can be converted to a surface term. The asymptotic forms (3.8) of γ_{ab} and γ'_{ab} now imply that the integrand of the surface term in fact vanishes identically (at infinity). Consequently, (3.7) reduces just to:

$$(\omega)_g(\gamma,\gamma') = \frac{1}{4k}\int_\Sigma(\gamma_{ab}\nabla_n\gamma'_{cd} - \gamma'_{ab}\nabla_n\gamma_{cd})\,\epsilon^{anc}{}_q\,\mathrm{d}v^{bdq}. \tag{3.9}$$

This is our final expression of the presymplectic 2-form ω on the phase space Γ. Apart from overall numerical factors, this is also the expression used in Refs. 7-13.

Given this expression, two questions arise immediately: Do the boundary conditions ensure that the integral is finite? Do they imply that it is independent of one's choice of the Cauchy surface Σ? For the phase space description to be meaningful, it is of course critical that these questions be answered affirmatively. Let us begin with the first. If one just assumes —as is often done— that the metrics g_{ab} in Γ should approach the flat background η_{ab} as $1/\rho$, one finds that the integrand in (3.9) is guaranteed to fall off only as $1/\rho^3$, so that the integral defining ω would, in general, diverge logarithmically. It is precisely to ensure that this does *not* occur that we need to be careful with the boundary conditions. What is the situation with the conditions we imposed in section 3.1? Choose for Σ the hypersurface given (possibly outside some compact set) by t=0. Then, the integral in (3.9) can be expressed as:

$$\omega(\gamma,\gamma') = \frac{6}{k}\int_R^\infty r^{-1}\mathrm{d}r\oint_{S^2}(\delta\sigma\,\partial_r\delta'\sigma - \delta'\sigma\,\partial_r\delta\sigma)\,\mathrm{d}^2v \\ + \text{manifestly finite terms}, \tag{3.10}$$

for any "radius" R outside which Σ is topologically $\mathbb{R}^3$. In this simplification, we have already used conditions (i) and (iii) of section 3.1 on the $1/\rho$-parts of γ_{ab} and γ'_{ab}. Yet, we now see explicitly that in the absence of a careful adjustment of boundary conditions the integral does, in general, diverge logarithmically. However, the parity conditions we imposed on σ (condition (ii) of section 3.1) come to rescue. Because of these conditions, the 2-sphere integral in (3.10) vanishes identically, thereby rendering the whole integral finite.† Note however that, since we used conditions (i) and (iii) to arrive at (3.10), the parity condition (ii) alone would not have assured finiteness; all three conditions on the $1/\rho$ part of g_{ab} are needed in our proof.

Let us now consider the second question: Are the boundary conditions strong enough to ensure that the integral is independent of the choice of the Cauchy surface Σ? Since g_{ab} is Ricci-flat and since γ_{ab} and γ'_{ab} satisfy the

† Thus, the integral (3.9) defining ω is in fact defined as follows: First perform the integral on a subset of Σ defined by $r \le r_0$ and then take the limit as r_0 tends to infinity. The following simple example illustrates the necessity of this subtlety. Consider the function $f(r,\theta) = (\sin\theta)/r^2$ on $\mathbb{R}^2$. The integral of this function over the region of $\mathbb{R}^2$ defined by $r \ge R > 0$ appears to diverge logarithmically. Yet if we first perform the integral up to $r = r_0 \ge R$ and then take the limit as r_0 tends to infinity, we get a finite answer, namely zero.

linearized Einstein equation off g_{ab},

$$2\,\nabla^m\nabla_{(a}\gamma_{b)m} - \nabla_a\nabla_b\gamma - \nabla^m\nabla_m\gamma_{ab} = 0 \tag{3.11}$$

(where γ is the trace of γ_{ab}), it follows either by an explicit calculation or from the general arguments given in section 2 that the "current" defining ω in (3.9) is divergence-free. Hence, given any two Cauchy surfaces Σ and Σ' bounded by a timelike tube $\mathcal{T}_\infty$ at infinity, the difference between the value of ω evaluated on Σ and on Σ' is given by the integral:

$$\frac{1}{4k}\int_{\mathcal{T}_\infty}(\gamma_{ab}\nabla_n\gamma'_{cd} - \gamma'_{ab}\nabla_n\gamma_{cd})\,\epsilon^{anc}{}_q\,\mathrm{d}v^{bdq}. \tag{3.12}$$

Thus, the surface independence of ω is guaranteed if and only if this last integral vanishes. If Σ and Σ' are related to one another by an asymptotic time translation, the integral can be seen to vanish just by a power counting argument: the integrand falls off as $1/\rho^3$, while the volume of integration grows only as ρ^2. However, if the two surfaces are relatively boosted, this simple argument is insufficient. It is here that the boundary conditions once again enter in a delicate way. Let us consider a set of hypersurfaces $\mathcal{T}_\rho \subset \mathcal{H}_\rho$, where $\mathcal{H}_\rho$ is the $\rho = const$ hyperboloid, which tend to $\mathcal{T}_\infty$ as ρ tends to infinity. Then, because of the asymptotic form (3.8) of γ_{ab} and γ'_{ab}, (3.12) reduces to an expression of the form $\lim_{\rho\to\infty}\int_{\mathcal{T}_\rho} O(1/\rho^4)\,\mathrm{d}^3v$, which vanishes, since, even when Σ and Σ' are relatively boosted, the volume of $\mathcal{T}_\rho$ grows only as ρ^3. In this calculation, it is the condition (iii) which ties the "angular-angular" component of the leading order piece of g_{ab} to its "ρ-ρ" component that plays the key role.

To summarize then, using the first order, Palatini form of the action, we have obtained the expression of the presymplectic 2-form ω on Γ. Our boundary conditions at spatial infinity are sufficient to ensure that the 2-form is well-defined and that it is independent of the choice of the Cauchy surface Σ used in its evaluation.

3.3 *The kernel of* ω

As we saw in section 2, in general the presymplectic 2-form has a kernel; the degenerate directions are to be interpreted as infinitesimal gauge motions. In general relativity, one would therefore expect that the degenerate directions of ω are intertwined with space-time diffeomorphisms. We shall now see that there is indeed a precise sense in which this is true. Furthermore, since we are using a manifestly covariant framework, unlike in the standard Hamiltonian formulations based on initial data on spacelike surfaces, we will be able to treat all space-*time* diffeomorphisms on the same footing.

Let ξ^a be a smooth vector field on the 4-manifold M. Under the action of the diffeomorphisms it generates, each metric g_{ab} undergoes an infinitesimal change, given by $\gamma_{(\xi)ab} := \mathcal{L}_\xi g_{ab}$. Since g_{ab} has vanishing Ricci tensor, each of the 1-parameter family of metrics obtained by the action, on g_{ab}, of the 1-parameter group of diffeomorphism generated by ξ^a is also Ricci flat. Hence (or, by explicit calculation) it follows that, for all ξ^a, the tensor field $\gamma_{(\xi)ab} \equiv 2\nabla_{(a}\xi_{b)}$ satisfies the linearized Einstein equation (3.11). Thus, each vector field ξ^a on M gives rise to a vector field $\gamma_{(\xi)ab}$ on the phase space Γ (provided of course $\gamma_{(\xi)ab}$ satisfies the boundary conditions (3.8)). Based on the intuitive idea that diffeomorphisms are the gauge transformations of general relativity, one would expect these vector fields to belong to the kernel of the presymplectic structure ω.

Let us examine if this is indeed the case. Let γ_{ab} be an arbitrary vector field on Γ. Then, a straightforward but rather long calculation yields:

$$\omega(\gamma, \mathcal{L}_\xi g) = -\frac{1}{2k}\left[\oint_{\partial\Sigma}\epsilon^{mn}{}_{pq}\,\xi^p\nabla_n\gamma_{ms}\,\mathrm{d}v^{qs} + \oint_{\partial\Sigma}\epsilon^{mnp}{}_q\,\epsilon^{qabc}\,t_c\gamma_{mb}\nabla_{[a}\xi_{p]}\,\mathrm{d}S_n\right]. \tag{3.13}$$

Thus, we see that, for *any* ξ^a for which $\gamma_{(\xi)ab} \equiv \mathcal{L}_\xi g_{ab}$ satisfies the boundary conditions (3.8), the symplectic inner product between an *arbitrary* vector field γ_{ab} and $\mathcal{L}_\xi g_{ab}$ depends only on the *asymptotic* values of ξ^a, its derivatives, and γ_{ab}. This remarkable behavior occurs precisely because the metric g_{ab} and the linearized field γ_{ab} satisfy the field equations. However, we note that, in general, the integral in (3.13) does *not* vanish. Thus, the covariant framework tells us that not all diffeomorphisms are to be regarded as gauge! In fact, it is quite straightforward to show that the integral vanishes for all g_{ab} and γ_{ab} if and only if the vector field ξ^a vanishes at infinity. Thus, it is precisely those diffeomorphisms which (preserve the boundary conditions and which) are the identity at infinity that are to be regarded as gauge. The general argument given in section 2 now tells us that these degenerate directions are integrable and that we can pass on to the reduced phase space $\hat{\Gamma}$ by quotienting Γ by their integral surfaces. $\hat{\Gamma}$ naturally inherits a (non-degenerate) symplectic structure $\hat{\omega}$ from ω. Points of the symplectic manifold $(\hat{\Gamma}, \hat{\omega})$ represent the "true degrees of freedom" of the asymptotically flat gravitational field. Finally, we recall that the asymptotic symmetry group can be defined as the quotient of the group of space-time diffeomorphisms which preserve the boundary conditions by its subgroup consisting of the asymptotically identity diffeomorphisms. For our choice of boundary conditions, this is the Poincaré group.[19] The covariant phase space formulation tells us that the action of these asymptotic symmetries is *not* to be regarded as gauge. In particular, this action can be unambiguously projected down to the reduced phase space, $\hat{\Gamma}$.†

What is the structure of the gauge group of the theory? It is straightforward to check that the mapping $\xi^a \to \gamma_{(\xi)}$ from the space of vector fields on M to the space of vector fields on Γ is linear and $1-1$, and that the Lie bracket on Γ between $\gamma_{(\xi)}$ and $\gamma_{(\bar{\xi})}$ is precisely the vector field $\gamma_{([\xi,\bar{\xi}])}$ associated with the Lie bracket $[\xi, \bar{\xi}]$ on M between ξ^a and $\bar{\xi}^a$. Therefore, we have a faithful representation on Γ of the group of diffeomorphisms that preserve our boundary conditions on M. Furthermore, using the relation between this covariant phase space and the standard phase space of the $3+1$ framework, one can show[20] that the vector fields $\gamma_{(\xi)}$ exhaust the kernel of ω. Thus, the gauge group on Γ is precisely the image under the map $\xi \to \gamma_{(\xi)}$ of the group of all diffeomorphisms which (preserve our boundary conditions and) are asymptotically identity.

How does this situation compare with that in the standard[16] $3+1$ phase space formulation? In that formulation, constraints, being first class, generate gauge. However, the Poisson algebra of constraints contains structure functions rather than structure constants; it is open in the BRST sense. Consequently, the structure of the gauge group in that theory does not have a simple relation with that of the space-*time* diffeomorphism group.[15] However, it is nonetheless true that each asymptotically identity diffeomorphism is represented as a gauge motion in that framework as well, in the sense that it is generated by constraints. Finally, note that whereas

† This $\hat{\Gamma}$ does not have the conical singularities that arise in the analogous construction in the spatially compact case. This is because all global Killing fields belong to the asymptotic symmetry —rather than gauge— group; their action is not factored out in the quotient construction leading to $\hat{\Gamma}$.

in Yang-Mills theories, gauge transformations correspond to motions in the internal space, in general relativity they correspond to motions in space-time. Therefore, there is a difference in the sense in which the term diffeomorphisms are regarded as gauge motions only in the sense that, to specify a 4-geometry, it is sufficient to provide a one-parameter family of initial data sets which are related to one another by, say, an asymptotic time-translation; it is redundant to provide, in addition, the data sets which are related to these by diffeomorphisms which are asymptotically identity. Thus, as in Yang-Mills theories, one can use a gauge fixing procedure and select, from each equivalence class of initial data related to each other by asymptotically identity diffeomorphisms, one representative and describe dynamics in terms of the evolution of these representatives. In the covariant phase space formulation, the situation is perhaps simpler: We say that two metrics g_{ab} and $\bar{g}_{ab}$ on M are gauge related if one is the image of the other under an asymptotically identity diffeomorphism. (Note that, if we just perform a diffeomorphism and do not carry the space-time metric with it, the motion is *not* to be regarded as gauge.)

4 THE ADM 4-MOMENTUM

In section 3.3, we saw that if a vector field ξ^a on M represents an asymptotic symmetry —*i.e.* a generator of the asymptotic Poincaré group— the corresponding vector field $\gamma_{(\xi)} \equiv \mathcal{L}_\xi g_{ab}$ does *not* belong to the kernel of ω. Nonetheless, from our discussion in section 2 (see remark 5 at the end of that section), it does follow that $\gamma_{(\xi)}$ is a canonical transformation; $\mathcal{L}_{\gamma_{(\xi)}}\omega = 0$. One can therefore ask what its generating function is. From our experience with Hamiltonian mechanics we know that the generators of translations can be interpreted as the energy-momentum of the system, and those of Lorentz transformations as the relativistic angular momentum. In general relativity, we already know the correct expressions for these physical quantities. Therefore, the question now arises if the generating functions of asymptotic symmetries computed by Hamiltonian methods agree with these expressions we already have from geometric considerations.[20,21] Note that the answer to this question is not obvious, especially because now the translational and Lorentz-rotational symmetries are only asymptotic and not exact. Indeed, a detailed calculation is needed, a calculation which depends on the precise expression of the presymplectic structure and the specific boundary conditions we used in the construction of the covariant phase space. In this section, we shall illustrate how these calculations are carried out by explicitly showing that the generating function of asymptotic translations on the covariant phase space Γ are precisely the components of the ADM 4-momentum.

Recall that, by definition, the generator H of the infinitesimal canonical transformation $\gamma_{(\xi)} \equiv \mathcal{L}_\xi g_{ab}$ must satisfy the equation

$$\begin{aligned}\omega(\gamma, \mathcal{L}_\xi g) &= \mathrm{i}_\gamma \mathrm{d} H_\xi \equiv \mathcal{L}_\gamma H_\xi \\ &\equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(H_\xi[g+\epsilon\gamma] - H_\xi[g]\right),\end{aligned} \tag{4.1}$$

for arbitrary vector fields γ on Γ. This equation determines H uniquely up to an additive constant. Therefore, our strategy is to consider the case in which ξ^a is an asymptotic translation, evaluate the left hand side of (4.1) explicitly, and then solve the differential equations we thus obtain on H_ξ. For a general (boundary conditions preserving) vector field ξ^a, the left hand side was already computed in section 3, and the result given in (3.13) as a sum of two surface integrals. For the case of translations, the second surface integral vanishes identically (simply by power counting). Furthermore, in the first integral, we can replace $\epsilon^{mn}{}_{pq}$, the metric g_{ab} and the derivative operator ∇ by their asymptotic values. With these simplifications, (4.1) now becomes,

$$-\frac{1}{2k}\oint_{\partial\Sigma} \mathring{\epsilon}^{mn}{}_{pq}\, \mathring{\xi}^p\, \partial_n \gamma_{ms}\, \mathrm{d}\mathring{v}^{qs} = \mathcal{L}_\gamma H_\xi, \tag{4.2}$$

where zero over fields denotes their asymptotic values (determined by the flat background metric η_{ab}). Since this equation is linear in γ_{ab}, using the interpretation of γ_{ab}, it can be readily integrated. The solution is simply:

$$H_\xi = -\frac{1}{2k}\oint_{\partial\Sigma} \mathring{\epsilon}^{mn}{}_{pq}\, \mathring{\xi}^p\, \partial_n g_{ms}\, \mathrm{d}\mathring{v}^{qs}. \tag{4.3}$$

Of course, the differential equation (4.2) can determine the Hamiltonian $H_x i$ only up to an additive constant. As usual, we eliminate this freedom by requiring that the value of the Hamiltonian at Minkowski space be zero. Then (4.3) is the unique solution to (4.2).

Let us now use the asymptotic form of g_{ab} to further simplify this expression. Using (3.2), we have:

$$H_\xi = -\frac{1}{k}\oint_{\partial\Sigma} \mathring{\epsilon}^{mn}{}_{pq}\, \xi^p\, \rho^{-2}\partial_m(\sigma\rho)\, \sigma_{ns}\, \mathrm{d}\mathring{v}^{qs}, \tag{4.4}$$

where σ_{ab} is the induced metric on the 2-sphere intersection of Σ with the constant ρ hyperboloids. Using the intrinsic geometry of the unit hyperboloid, it is straightforward to reduce (4.4) to the "canonical" form of the ADM 4-momentum one finds in the geometrical, 4-dimensional approaches[20,21] to conserved quantities:

$$H_\xi = \frac{1}{2k}\oint_{\partial\Sigma} \epsilon^m{}_{qs}\, \xi^p (D_m D_p \sigma + \sigma h_{mp})\, \mathrm{d}v^{qs}, \tag{4.5}$$

where ϵ_{mqs} and D are, respectively, the alternating tensor and the derivative operator compatible with the unit hyperboloid metric h_{ab}. Finally, we note that both the generator H_ξ and the canonical transformation it generates can be projected down unambiguously to the reduced phase space $(\hat\Gamma, \hat\omega)$. A point on the reduced phase space is an entire equivalence class of histories — two histories being equivalent if they are gauge related— and asymptotic translations map one class to another. The flow corresponding to an asymptotic time translation can be regarded as dynamics.

How does this calculation compare to its analog in the 3+1 frameworks? There are two key differences. In the covariant calculation, the Hamiltonians generating asymptotic translations are made up *entirely* of surface terms. Secondly, given an asymptotic translation ξ^a, its generating function or Hamiltonian H_ξ is unique. However, since the presymplectic structure is degenerate, a given Hamiltonian does *not* give rise to a unique Hamiltonian vector field; we have the freedom to add to the Hamiltonian vector field any vector field which is everywhere in the kernel of ω. In the 3+1 frameworks, on the other hand, the Hamiltonians generating asymptotic translations contain a volume term —integrals of constraints smeared out with the appropriate lapses and shifts— in addition to the surface terms. Thus, although, on the constraint surface, the numerical values of these Hamiltonians coincide with their analogs in the covariant framework, the functional forms of the two sets are quite different. In particular, in the 3+1-framework, the expression of the Hamiltonian depends on the values of the lapse and shift fields —and hence of the asymptotic translation ξ^a— *in the interior*. In the covariant framework, on the other hand, the expression is insensitive to the way in which the asymptotic translation is extended to the interior. Secondly, on the $3+1$ phase space, the symplectic structure is non-degenerate and each Hamiltonian generates precisely one vector field.

Remark: It is easy to show that our Hamiltonians are numerically equal to those in the $3+1$ framework (if one restricts oneself there to the constraint surface). As an example, let us consider the case when ξ^a is an asymptotic time translation and use the form (4.3) of the generating function, where the surface of integration is chosen to be orthogonal to ξ^a. Then, we have: $\overset{\circ}{\epsilon}{}^{mn}{}_{pq}\xi^p \, \mathrm{d}\overset{\circ}{v}{}^{qs} \sim -(\overset{\circ}{q}{}^{mp}\overset{\circ}{q}{}^{ns} - \overset{\circ}{q}{}^{ms}\overset{\circ}{q}{}^{np})\,\mathrm{d}\overset{\circ}{S}_p$, where $\overset{\circ}{q}_{ab}$ is the metric defined on Σ by η_{ab}. Therefore, we obtain

$$H_\xi = \frac{1}{2k}\oint_{\partial\Sigma}(\partial_n g_{ms} - \partial_m g_{ns})\,\overset{\circ}{q}{}^{mp}\overset{\circ}{q}{}^{ns}\,\mathrm{d}\overset{\circ}{S}_p = E_{\mathrm{ADM}}, \tag{4.6}$$

the standard form of the ADM energy associated with g_{ab}.

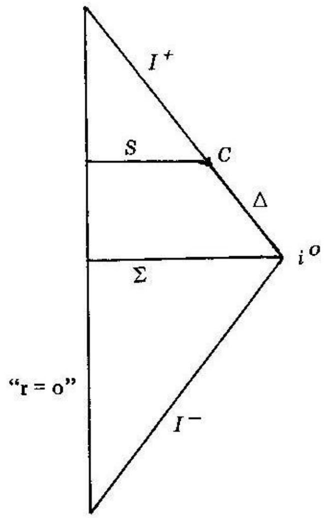
5 THE BONDI 4-MOMENTUM

Let us now consider space-times which are asymptotically flat also at null infinity and examine the action of BMS translations[17] on the corresponding phase space Γ. Our experience with the boundary conditions at spatial infinity suggests that this action would also preserve the presymplectic structure and that the corresponding generating functionals would provide us the expression of the Bondi 4-momentum. We shall now see that this general expectation is indeed correct. At spatial infinity, the ADM 4-momentum was in fact first introduced[16] using a Hamiltonian framework, albeit one based on a $3+1$ description. At null infinity, by contrast, the expression of the 4-momentum was initially introduced by Bondi and Sachs using only geometrical and physical intuition on the expected fall-off of the metric and the Weyl curvature as one recedes to infinity along null directions. It later turned out that the expression of the *flux* of 4-momentum carried away by the gravitational waves, obtained by Bondi and Sachs, could in fact be recovered from a phase space description of the radiative modes of the gravitational field.[10] The present covariant Hamiltonian treatment extends that result; it allows us to derive expressions of both the Bondi 4-momentum and its flux using phase space methods. We do not see any *a priori* reason why the

Hamiltonian expressions of the 4-momentum must coincide with those of Bondi and Sachs. The fact that they do provides additional motivation in support of the boundary conditions normally used at null infinity, and the gravitational radiation theory that results in the framework of exact general relativity.

We shall divide this discussion into three subsections. The first, 5.1, is devoted to preliminaries. We state the precise boundary conditions assumed at null infinity and summarize certain results, due to Geroch and Xanthopoulos,[22] on the asymptotic behavior of linearized fields in space-times satisfying these conditions. In 5.2, we recall the structure of the asymptotic symmetry group —particularly its translation subgroup— as well as the definition of the Bondi 4-momentum. In the final part, 5.3, we present the main result (see figure): Given *any* cross-section $\mathcal{C}$ of future null infinity, $\mathcal{I}^+$, the sum of the Bondi 4-momentum evaluated at $\mathcal{C}$ and the total flux of momentum through the portion of $\mathcal{I}^+$ in the past of $\mathcal{C}$ is the generator, on Γ, of the BMS translations.

The Penrose diagram of an asymptotically flat space-time ("r-t plane"). Σ is a Cauchy surface; I^+ represents null infinity; S is a partial Cauchy surface that intersects I^+ in a 2-sphere cross section C; and, Δ is the part of I^+ in the past of C. In section 5 we use $S \cup \Delta$ in place of the Cauchy surface Σ to evaluate the presymplectic 2-from and the Hamiltonian generating translations.



5.1 Boundary conditions

As at spatial infinity, the boundary conditions to be used at null infinity are most succinctly expressed as requirements on a certain type of completion of the physical space-time. At spatial infinity, we avoided the completion because the ideas involved are rather recent.[19] At null infinity, on the other hand, the ideas were first introduced in the literature already in the mid-sixties[23] and it is well known that what is needed is a confomal completion. Therefore, in this section, we shall formulate the boundary conditions in terms of this completion.

A space-time (M, g_{ab}) will be said to be asymptotically flat at null infinity if there exists a manifold $\hat{M}$ with boundary $\partial M =: \mathcal{I}$ which is topologically $S^2 \times \mathbb{R}$, and a Lorentzian metric $\hat{g}_{ab}$ thereon such that: (i) on M, $\hat{g}_{ab}$ is conformally related to g_{ab}; $\hat{g}_{ab} = \Omega^2 g_{ab}$; (ii) at $\mathcal{I}$, we have $\Omega = 0$ and $\nabla_a \Omega \neq 0$. The vanishing of the conformal factor Ω at $\mathcal{I}$ ensures that $\mathcal{I}$ is "at infinity," while the fact that $\nabla_a \Omega$ does not vanish at $\mathcal{I}$ ensures that "Ω falls off as $1/r$." In general, the boundary $\mathcal{I}$ has two components, one, $\mathcal{I}^+$, serving as the future light cone of the point at spatial infinity, and the other, $\mathcal{I}^-$, serving as the past light cone. We shall focus on the future boundary, although the discussion applies in an obvious fashion also to the past boundary.

The phase space Γ will contain only vacuum solutions. For such asymptotically flat metrics g_{ab}, one can show that it is always possible to choose a conformal factor Ω such that $\hat{\nabla}^a \hat{\nabla}_a \Omega = 0$ in a neighborhood of $\mathcal{I}^+$ and $\hat{\nabla}_a \hat{\nabla}_b \Omega = 0$ *at* $\mathcal{I}^+$. *Throughout this section we shall assume that such a choice has been made.* The field equations also imply that $\mathcal{I}^+$ is a null 3-surface, *i.e.* its normal $n^a := \hat{g}^{ab} \nabla_b \Omega$ is null. $\mathcal{I}^+$ is "ruled" by the integral curves of this vector field n^a. The integral curves are called *generators* of $\mathcal{I}^+$. The space of these generators, called the *base space*, is topologically S^2. The intrinsic "metric" on $\mathcal{I}^+$, q_{ab}, is degenerate with signature $(0, +, +)$; it is the lift to $\mathcal{I}^+$ of the metric on the base space. Since the 2-sphere admits a unique conformal class of metrics, it is possible to further adjust the conformal factor Ω such that the metric q_{ab} on the base space is precisely the standard, unit 2-sphere metric. Such a "conformal frame" is called a *Bondi-frame*. (Note, incidentally, that this frame is not unique; the 2-sphere admits distinct, conformally related metrics each of which is globally isometric with the unit 2-sphere metric.)

Finally, let us consider the asymptotic behavior of linearized solutions, *i.e.* of tangent vectors to Γ. Geroch and Xanthopoulos[22] have analysed this issue in detail; here we will only state the results we need. Fix an asymptotically flat, vacuum space-time, (M, g_{ab}) and consider on it a linearized solution γ_{ab} to Einstein's equation which has initial data of compact support on some Cauchy surface. Set $\tau_{ab} := \Omega^{-1}\gamma_{ab}$, $\tau_a := \Omega^{-1}\tau_{ab} n^b$ and $\tau := \hat{g}^{ab}\tau_{ab}$. Then, possibly after a gauge transformation, γ_{ab} satisfies the following conditions: (i) in a neighborhood of $\mathcal{I}^+$, $\hat{\nabla}^b \tau_{ab} - \hat{\nabla}_a \tau - 3\,\tau_a = 0$; (ii) τ_{ab} and τ_a admit smooth limits at $\mathcal{I}^+$. One can show[9,10] that the trace-free part of the pull-back to $\mathcal{I}^+$ of τ_{ab} captures precisely the two radiative modes of the gravitational field contained in γ. At the level of the functional analytic rigor of this paper, we can therefore say that the tangent space to Γ at the point g_{ab} is spanned by the linearized fields γ_{ab} satisfying the Geroch-Xanthopoulos conditions.

5.2 *The asymptotic symmetries and the Bondi 4-momentum*

To discuss asymptotic symmetries, let us begin, as before, by considering diffeomorphisms which preserve the boundary conditions. Detailed considerations show that the diffeomorphism generated by a vector field ξ^a on M preserves the boundary conditions if and only if ξ^a admits a smooth extension to $\mathcal{I}^+$ and $\Omega^2 \mathcal{L}_\xi g_{ab}$ vanishes on $\mathcal{I}^+$.[24] If the vector field ξ^a vanishes on $\mathcal{I}^+$, it leaves the structure of null infinity untouched and is therefore regarded, in these geometrical considerations, as "gauge." The asymptotic symmetry group —the Bondi-Metzner-Sachs (BMS) group— is the result of quotienting the space of all diffeomorphisms that preserve the boundary conditions by this "gauge group." Of special interest to us is its translation subgroup. At $\mathcal{I}^+$, consider first infinitesimal BMS transformations of the type $\xi^a \cong \alpha\, n^a$, with $\mathcal{L}_n \alpha \cong 0$, *where* $\cong$ *stands for "equality at* $\mathcal{I}^+$." These are called BMS supertranslations. Note that, under a conformal rescaling, $\Omega \to \mu\Omega$, we have $n^a \to \mu^{-1} n^a$ whence α transforms as $\alpha \to \mu\alpha$. Thus, α is a conformally weighted scalar with weight one. Now consider those BMS supertranslations for which, in a Bondi conformal frame, $q^{ab}\partial_b \alpha$ is a conformal Killing field of the unit 2-sphere metric q_{ab} on the base space of $\mathcal{I}^+$. It is straightforward to verify that this can occur if and only if α is a linear combination of the first four spherical harmonics. This is a 4-dimensional subspace of the space of BMS supertranslations. It turns out that this subspace is independent of the particular choice of the Bondi conformal frame made in its construction. This is the space of BMS translations. If we choose $\alpha \cong 1$ in a Bondi frame, the corresponding BMS symmetry is a time translation. Conversely, *every* time-translation can be represented by the vector field $\xi^a \cong n^a$ in *some* Bondi conformal frame. We will use these facts in the next subsection to simplify our calculations.

Let us now introduce the Bondi 4-momentum. Given a cross-section $\mathcal{C}$ of $\mathcal{I}^+$, the Bondi 4-momentum is a linear mapping from the space of BMS translations to the space of functions on the phase space Γ. It represents the 4-momentum "left over" at the retarded instant of time represented by the cross-section $\mathcal{C}$, after allowing for gravity waves to carry away the 4-momentum for all retarded times in the past of $\mathcal{C}$. Let us begin by introducing the physical fields that feature in its expression. First of all, because of asymptotic flatness, the Weyl tensor $\hat{C}^{abcd}$ of $\hat{g}_{ab}$ vanishes at $\mathcal{I}^+$, whence $K^{abcd} := \Omega^{-1}\hat{C}^{abcd}$ admits a smooth limit there.[23] $K^{ac} := K^{abcd}n_b n_d$ (with $n_a := \hat{\nabla}_a\Omega$) may be regarded as the "electric part of the Weyl tensor;" in stationary space-times, its value at $\mathcal{I}^+$ carries the full information about the 4-momentum of the corresponding isolated system. In the presence of gravitational radiation, however, the Weyl curvature does not suffice. We also need an additional field called the *Bondi news tensor*, N_{ab}. This tensor can be defined most succinctly in a Bondi conformal frame; *we will therefore restrict ourselves to a Bondi frame throughout what follows.* Set $\hat{S}_{ab} := \hat{R}_{ab} - \frac{1}{6}\hat{R}\,\hat{g}_{ab}$, where $\hat{R}_{ab}$ is the Ricci tensor of $\hat{g}_{ab}$, and denote its pull-back to $\mathcal{I}^+$ by s_{ab}. It is easy to show that the tensor field s_{ab}, thus defined intrinsically on $\mathcal{I}^+$, is normal to n^a. Hence $N_{ab} := s_{ab} - \frac{1}{2}s_{mn}q^{mn}\,q_{ab}$, where q^{ab} is *any* inverse of the degenerate intrinsic metric q_{ab} on $\mathcal{I}^+$, is just the trace-free part of s_{ab}. This is the Bondi news tensor. We can now define the Bondi 4-momentum associated with any cross-section $\mathcal{C}$ of $\mathcal{I}^+$. Given any BMS translation $\xi^a \cong \alpha n^a$, the corresponding component $P_{(\alpha)}$ of the Bondi 4-momentum is given by:

$$P_{(\alpha)}[\mathcal{C}] = \frac{1}{k}\oint_{\mathcal{C}} \left[\alpha\, K^{ab}l_a l_b + \tfrac{1}{2}\alpha\,(D_a l_b)\,N_{cd}\,q^{ac}q^{bd}\right] \mathrm{d}^2\hat{v}, \tag{5.1}$$

where l_a is the covector field on $\mathcal{I}^+$ satisfying $l_a n^a = 1$ which is orthogonal to the cross-section $\mathcal{C}$, D is the pull-back to $\mathcal{I}^+$ of the derivative operator $\hat{\nabla}$ compatible with the metric $\hat{g}_{ab}$, and, as before, q^{ab} is any inverse of q_{ab}. The positive energy theorems at null infinity ensure us that if we restrict ourselves to time translations, *e.g.* by setting $\alpha = 1$, this integral is always non-negative, being equal to zero if and only if the space-time is flat in the future of any partial Cauchy surface whose boundary is $\mathcal{C}$. The value of the integral, however, depends on the choice of the cross-section; there is leakage

of the 4-momentum through $\mathcal{I}^+$ because of gravitational radiation. Given any 3-dimensional region Δ within $\mathcal{I}^+$, the total flux $F_{(\alpha)}[\Delta]$ of the 4-momentum that has leaked through Δ is given by:

$$F_{(\alpha)}[\Delta] := \frac{1}{2k}\int_{\Delta} \left(D_a D_b\,\alpha + \tfrac{1}{2}\alpha N_{ab}\right) N_{cd}\,q^{ac}q^{bd}\,\mathrm{d}^3\hat{v}. \tag{5.2}$$

Again, if we restrict ourselves to time translations by setting $\alpha = 1$, we find that the flux of energy across Δ is positive. Finally, given any two cross-sections $\mathcal{C}$ and $\mathcal{C}'$ of $\mathcal{I}^+$ bounding a volume Δ (with, say, $\mathcal{C}'$ in the future of $\mathcal{C}$), the difference between the Bondi 4-momentum evaluated at $\mathcal{C}$ and $\mathcal{C}'$ is precisely its flux of through Δ:

$$P_{(\alpha)}[\mathcal{C}] - P_{(\alpha)}[\mathcal{C}'] = F_{(\alpha)}[\Delta]. \tag{5.3}$$

5.3 *Hamiltonians generating BMS translations*

With the preliminaries out of the way, we can now apply the covariant Hamiltonian framework. Our phase space Γ will now consist of vacuum metrics g_{ab} which are asymptotically flat both at spatial and null infinity. We will therefore be able to use all the results obtained in the last two sections. (One *can* consider metrics which are asymptotically flat only at null infinity, and rederive the null infinity analogs of these results. From physical considerations, however, it is natural to require asymptotic flatness in both regimes.) The idea now is to replace the Cauchy surface Σ used in the evaluation of the symplectic structure and generating functionals of canonical transformations in sections 3 and 4, by the union of a partial Cauchy surface S which intersects $\mathcal{I}^+$ in a cross-section $\mathcal{C}$ and the portion Δ of $\mathcal{I}^+$ which is in the past of $\mathcal{C}$ (see figure). Since $S \cup \Delta$ and Σ bound a 4-volume in the physical space-time, given any curl-free 3-form J_{abc} on M (which admits a smooth limit to $\mathcal{I}^+$ and remains bounded at the point at spatial infinity) we have:

$$\int_{\Sigma} J_{abc}\,\mathrm{d}\hat{v}^{abc} = \int_{S\cup\Delta} J_{abc}\,\mathrm{d}\hat{v}^{abc}. \tag{5.4}$$

Using this identity, the action (3.9) of the symplectic structure ω on any two tangent vectors γ_{ab} and γ'_{ab} can now be expressed as an integral over $S \cup \Delta$. This expression turns out to be:[9]

$$(\omega)_g(\gamma,\gamma') = \frac{1}{4k}\Big[\int_{S}(\gamma_{ab}\nabla_n\gamma'_{cd} - \gamma'_{ab}\nabla_n\gamma_{cd})\,\epsilon^{anc}{}_q\,\mathrm{d}v^{bdq} \tag{5.5}$$

$$- \int_\Delta q^{ac}q^{bd}\{(\tau_{ab} - \tfrac{1}{2}q^{mn}\tau_{mn}\,q_{ab})\,\mathcal{L}_n(\tau'_{cd} - \tfrac{1}{2}q^{ij}\tau'_{ij}\,q_{cd}) - (\tau'_{ab} - \tfrac{1}{2}q^{mn}\tau'_{mn}\,q_{ab})\,\mathcal{L}_n(\tau_{cd} - \tfrac{1}{2}q^{ij}\tau_{ij}\,q_{cd})\}\,\mathrm{d}^3\hat{v}\Big].$$

Note that the integrand in the first integral is identical to the one in (3.9); only the surface of integration has changed from Σ to S. The one in the second integral, on the other hand, is obtained by taking a limit of the integrand in (3.9) as one approaches $\mathcal{I}^+$ and by replacing the terms involving γ_{ab} and their covariant derivatives by those involving τ, using boundary conditions satisfied both by g_{ab} and γ_{ab}. Equation (5.5) is the starting point of our Hamiltonian analysis in the covariant phase space framework.

To begin with, let us note the results which go over unchanged from sections 3 and 4. Given any vector field ξ^a on M which preserves our boundary conditions, we can construct the vector field $\gamma_{(\xi)ab} \equiv \mathcal{L}_\xi g_{ab}$ on Γ. Using the presymplectic structure (5.5) and the boundary conditions, one can again show that $\gamma_{(\xi)}$ is in the kernel of ω if and only if the vector field ξ^a vanishes on $\mathcal{I}^+$. Thus, the geometric notion of gauge, discussed above, and the Hamiltonian notion of gauge again coincide. If ξ^a preserves the boundary conditions but does *not* vanish on $\mathcal{I}^+$, its value on $\mathcal{I}^+$ supplies us with a BMS vector field. The general arguments of section 2 tell us that the corresponding vector field $\gamma_{(\xi)}$ on Γ is an infinitesimal canonical transformation. Its generating function is the conserved quantity corresponding to the BMS symmetry singled out by ξ^a.

Let us now restrict ourselves to BMS translations. The simplest case is that of the time translation corresponding to $\alpha = 1$ in the given Bondi frame. Then, on $\mathcal{I}^+$, ξ^a is given just by $\xi^a \cong n^a$. To analyse the relation between the infinitesimal canonical transformation $\gamma_{(\xi)}$ on Γ and the Bondi 4-momentum, we use once again equation (4.1), relating $\gamma_{(\xi)}$ to its generating functional H_ξ. The first step is to compute the presymplectic inner product between $\gamma_{(\xi)}$ and an arbitrary tangent vector γ_{ab}. Using the Geroch-Xanthopoulos results quoted in subsection 5.1, the first integral in (5.5) becomes:

$$\omega_S(\gamma, \mathcal{L}_\xi g) = \frac{1}{2k}\oint_C \left[\mathcal{L}_n(\tau_a)\,l^a - \mathcal{L}_l(\tau_a n^a)\right]\mathrm{d}^2\hat{v}, \tag{5.6a}$$

while the second yields:

$$\omega_\Delta(\gamma, \mathcal{L}_\xi g) = -\frac{1}{2k}\int_\Delta (\mathcal{L}_n\tau_{ab})\,N_{cd}\,q^{ac}q^{bd}\,\mathrm{d}^3\hat{v} + \frac{1}{4k}\oint_C \tau_{ab}N_{cd}\,q^{ac}q^{bd}\,\mathrm{d}^2\hat{v}. \tag{5.6b}$$

To obtain the Hamiltonian, we need to integrate, on the phase space Γ, the equation: $\mathcal{L}_\gamma H_\xi(g) = \omega_S(\gamma, \mathcal{L}_\xi g) + \omega_\Delta(\gamma, \mathcal{L}_\xi g)$, for all tangent vectors γ. In the case of the ADM 4-momentum, this task was straightforward because the corresponding equation, (4.2), involved, apart from ξ^a and γ, only those fields which depend on the fixed, fiducial metric η_{ab} rather than the dynamical metric g_{ab}. Equation (5.6), on the other hand, does contain terms involving g_{ab}. Therefore, it is not straightforward to integrate it. However, since we are in fact interested in the relation between the Bondi 4-momentum and the Hamiltonian, let us evaluate the directional derivatives of $P_{(\alpha=1)}[\mathcal{C}]$ and $F_{(\alpha=1)}[\Delta]$ in the direction of γ and compare the results with (5.6). A straightforward calculation using, again, the Geroch-Xanthopoulos results yields:

$$\mathcal{L}_\gamma P_{(\alpha=1)}[\mathcal{C}] = \frac{1}{2k}\oint_C \left[(\mathcal{L}_n\tau_a)\,l^a - \mathcal{L}_l(\tau_a n^a) + \tfrac{1}{2}\,\tau_{ab}N_{cd}\,q^{ac}q^{bd}\right]\mathrm{d}^2\hat{v}, \tag{5.7a}$$

and

$$\mathcal{L}_\gamma F_{(\alpha=1)}[\Delta] = -\frac{1}{2k}\int_\Delta (\mathcal{L}_n\tau_{ab})\,N_{cd}\,q^{ac}q^{bd}\,\mathrm{d}^3\hat{v}. \tag{5.7b}$$

Comparing equations (5.6) and (5.7), it is clear that, modulo an additive constant, the Hamiltonian H_ξ we are seeking is given precisely by:

$$H_\xi(g) = P_{(\alpha=1)}[\mathcal{C}] + F_{(\alpha=1)}[\Delta], \tag{5.8}$$

for any choice of the Bondi conformal frame (and of cross-section $\mathcal{C}$ of $\mathcal{I}^+$). Once again, we eliminate the freedom to add a constant to H_ξ by requiring that its value at the Minkowskian metric be zero. Since (5.8) holds in *any* Bondi frame, and since the Hamiltonian H_ξ, the Bondi 4-momentum $P_{(\alpha)}[\mathcal{C}]$, and the flux integral $F_{(\alpha)}[\Delta]$ are all linear in the BMS translation, it follows that equality (5.8) in fact holds for *all* BMS translations. This establishes our main result. The fact that the result is independent of the choice of the cross-section $\mathcal{C}$ of $\mathcal{I}^+$ can be considered as a reflection, in the Hamiltonian framework, of the identity (5.3).

Finally, note that in space-times which are asymptotically flat at both null and spatial infinity considered in this section, one can choose vector fields ξ^a which provide us with asymptotic translations in both regimes. The generating functional on the phase space Γ of the canonical transformation $\gamma_{(\xi)}$ can then be evaluated either using a Cauchy surface Σ or the union $S \cup \Delta$

of a partial Cauchy surface S and a portion Δ of $\mathcal{I}^+$. In the first case, the generating functional emerges as the ADM 4-momentum, and in the second, as the sum of the Bondi 4-momentum evaluated at C and the total 4-momentum carried away by gravitational radiation through the region Δ. Thus, we now have a Hamiltonian argument for the equality between these quantities. In the literature, there already exists a proof of this result.[18] Furthermore, that proof is rigorous; it involves only differential geometry and thus avoids the gaps in the present treatment of functional analysis entirely. However, from a physical viewpoint, the present Hamiltonian argument is perhaps more satisfactory. For, it brings out the essential reason behind the equality: both quantities are generators of asymptotic translations.

6 DISCUSSION

In the last three sections, we have seen how the covariant Hamiltonian framework can be used to derive useful results in general relativity, results which are outside the scope of the more familiar 3 + 1 phase space formulations of the theory. The framework clarifies the role of the space-*time* diffeomorphism group, enables one to show that the physical quantities introduced by Bondi and Sachs at null infinity do have a Hamiltonian basis, and establishes the relation between these quantities and those defined at spatial infinity. Can the framework be used to establish further new results? It does seem that it is ideally suited to shed light on a long standing question, that of the "correct" definition of angular momentum at null infinity. The question is complicated because, in the presence of gravitational radiation, it becomes impossible to reduce the BMS group to the Poincaré group, and the notion of angular momentum acquires the so-called supertranslation ambiguities. Over the years, a number of *inequivalent* definitions of angular momentum have been introduced. However, it was slowly realized that most definitions had certain physically undesirable properties and by now only one, due to Dray, Streubel and Shaw[25] has survived the viability criteria that one can write down on physical grounds. This definition is introduced by combining in an ingenious way some techniques from the symplectic geometry of radiative modes of the gravitational field and from Penrose's twistor theory. However, the physical basis of this procedure is still somewhat obscure. It

is likely that the definition can in fact be derived from the covariant phase space introduced here. If so, its status would improve substantially.

Thus, it does appear that the framework is likely to have a number of applications to classical general relativity and more generally, classical field theories. What is the status of applications to quantum theory? After all, the Hamiltonian formalism is often used as the point of departure in certain quantization procedures. For linear field theories, the framework has already been applied successfully (see *e.g.* Ref. 6). In the non-linear case, on the other hand, there are reasons to believe that the framework would not be as successful. Consider, for example, general relativity. Can one carry out a quantization of the covariant phase space, using *e.g.* techniques from geometric quantization? With this application in mind, the standard geometric quantization procedure has indeed been modified to accommodate the degenerate directions of the presymplectic structure.[26] What is needed now, is a certain "constrained polarization" which incorporates within it the "gauge directions." However, it has proved to be difficult to introduce explicitly the necessary structure. It is not yet clear if this is just a technical difficulty or a reflection of a deeper problem with this approach. Let us therefore suppose that this is only a technical problem and can somehow be surmounted. Even in that case, the resulting quantum theory would be rather uninteresting. For, in order to have a globally well-defined Hamiltonian vector field, one would restrict oneself right from the beginning to a phase space Γ which contains only "weak" 4-dimensional gravitational fields — Γ would have to be just a neighborhood of Minkowski space-time in an appropriate function space. Quantum mechanics on such a phase space would not reveal answers to the interesting problems such as the role of quantum effects near singularities; the phase space would simply not admit any! If one were to allow strong fields and space-time singularities, the Hamiltonian vector fields would be incomplete. It is then likely that quantum evolution would lead to the same singularities as the classical theory. Thus, roughly, if one were to carry out quantization using the covariant phase space of general relativity, the dynamical behavior of the resulting quantum theory would be essentially governed by that of the classical theory. How do the standard quantization procedures avoid this fate in quantum field theory? The reason can be intuitively understood as follows. In the standard description of quantum fields, configurations that do not satisfy the classical field equations have an important role to play. In a quantization scheme based on the covariant phase space, on the other hand, such configurations have no natural role. Even in simple systems with a finite number of degrees of freedom, these limitations are apparent. In the geometric quantization language, quantum wave functions are "suitably polarized" functions on the classical phase space. If the phase space consists of classical solutions, there would be no natural avenue for the genuinely non-classical effects such as quantum tunneling to occur, effects which we regard as the hallmark of quantum mechanics. It is very surprising indeed that the Hamiltonian description that seems so well suited for analysing physical problems of the classical theory should fare so poorly in the quantum theory. Is there a deep reason underlying this dichotomy?

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200 Years after Lagrange
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THE LAGRANGIAN APPROACH TO CONSERVED QUANTITIES IN GENERAL RELATIVITY

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In this paper we review general ideas and results concerning the geometric "Lagrangian" approach to conserved currents in field theories, showing how these results apply to General Relativity and to relativistic field theories.

1. INTRODUCTION

Although General Relativity is a well established theory of gravity interacting with external matter, after more than seventy years there is yet no general agreement on the definition of mass and, more generally, of conserved quantities associated to the gravitational field itself. This open problem was neatly pointed out by R. Penrose in the introduction to his paper [81] (see also [95]), whereby he