Detecting submanifolds of minimum volume with calibrations

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Contents

It is not easy to choose the topics of a course when the audience is not homogeneous. I will review some subjects which may be well known to some participants. With goodwill, the review could be considered a crash course for those students who have not seen the issues before.

I will try to be sparse with the notation, for instance I have managed without the exterior product. I will try to concentrate on the main ideas. I will not give many details and cheat a little, mostly avoiding orientation issues (easy but time consuming).

I will make some very informal comments, at risk of oversimplification.

I take the opportunity to include subjects that are central in Mathematics, such as octonions and the Hopf map from the 3- to the 2-sphere.

If you have not taken a course on smooth manifolds yet: When I say M is a submanifold of N, you can think of a curve M in a surface N or a surface M in $\mathbb{R}^3 = N$ (or higher dimensional analogues).

Homologically volume minimizing submanifolds

A calibration is a tool which allows us to prove that a submanifold M of a Riemannian manifold N has minimum volume among all submanifolds of N homologous to M.

Notation:

1-volume = length, 2-volume = area, 3-volume = usual volume

I will write k-volume or simply volume if k is understood from the context.

I do not have time to give the definition of homology class. I will give some examples that will help getting the feeling of it, I hope.

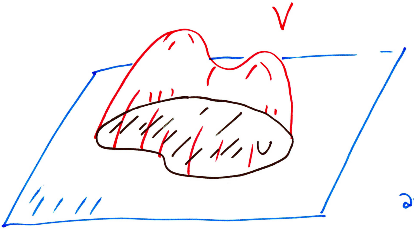
Definition

Let M be a closed oriented m-dimensional submanifold of N. One says that M is volume minimizing in N if for any open subset U of M with compact closure and smooth border ∂U (possibly empty), one has that

$$\mathrm{vol}_m(U) \leq \mathrm{vol}_m(V)$$

for any submanifold V of N of dimension m with compact closure and $\partial V = \partial U$.

Moreover, M is said to be homologically volume minimizing in N if in addition V is required to be homologous to U.

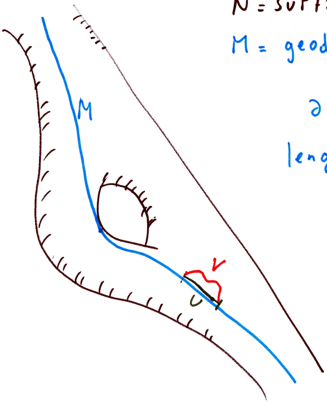


$N = \mathbb{R}^3$

$M = \text{plane}$

$\partial U = \partial V$

$\text{area}(U) < \text{area}(V)$

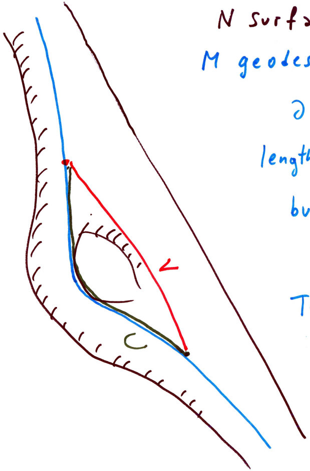


N = surface

M = geodesic curve in N

$$\partial U = \partial V$$

$$\text{length}(U) < \text{length}(V)$$



N surface

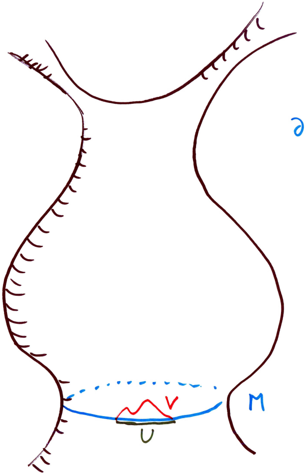
M geodesic curve in N

$\partial U = \partial V$

length U > length V

but U and V are not homologous

Then M is homologically length minimizing in N

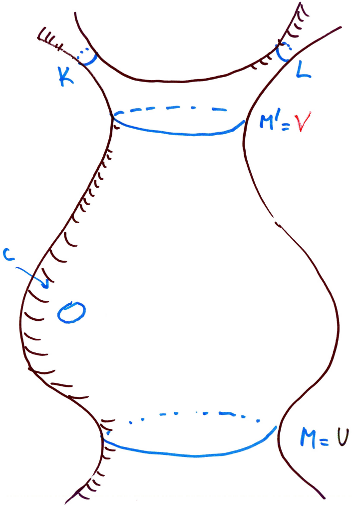


N surface

M circle

$\partial U = \partial V$, length U < length V

We look for $\overline{M}$ with minimum length in the homology class of M in N



M and M' are homologous and have the same border $= \phi$

length M' $\leq$ length M

K, L and C not homologous to M

$\bar{M}$ with minimum length in the homology class of M in N is $K \cup L$.

We actually need oriented submanifolds. I am cheating with that.

Remarks

a) Finding manifolds of minimum volume means looking for the minimum of a functional on an infinite dimensional space.

b) Advantage of calibrations over the calculus of variations. The latter provides only local minima.

Alternating multilinear algebra

$\det : \mathbb{R}^{n\times n} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is n-linear and alternating

$$\det (u_1, \ldots, u_n) = \pm \,\text{volume } (P) \,,$$

where P is the parallelepiped determined by $u_1, \ldots, u_n$.

The sign $\pm$ gives a notion of orientation.

Comments

If only one side of a parallelogram is doubled, the area of the parallelogram is doubled: $\det (2u_1, u_2) = 2 \det (u_1, u_2)$

A parallelepiped whose edges lie on a plane is degenerated and has zero volume: $\det (u, v, w) = 0$ if $\{u, v, w\}$ is l.d.

Alternating bilinear form:

$$\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

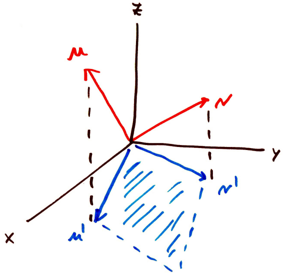
ω is bilinear and satisfies $\omega(u, u) = 0$ for any $u \in \mathbb{R}^n$.

Example

For $1 \leq i < j \leq n$, define

$$e^{i,j}(u, v) = \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix}$$

(project u, v onto the i-j wall of $\mathbb{R}^n$ and take determinant).



$$e^{1,2}(u,v) = \det(u', v')$$

Any alternating bilinear form $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be written as

$$\omega = \sum_{1 \le i < j \le n} a_{ij} e^{i,j}$$

for some $a_{ij} \in \mathbb{R}$. With respect to another basis the expression for ω may have fewer terms.

A bilinear alternating form assigns to the pair u, v a multiple of the area of the parallelogram determined by u, v, in a consistent way.

Analogous statements hold for alternating m-linear forms. We call then simply m-forms.

An example of a 3-form ω on $\mathbb{R}^5$: $\omega = 2e^{123} - e^{125} + e^{345}$

Definition

An *m-calibration ω on $\mathbb{R}^n$* is an m-form on $\mathbb{R}^n$ such that for any m-dimensional subspace W of $\mathbb{R}^n$ one has that

$$|\omega(u_1, \ldots, u_m)| \leq 1 \tag{1}$$

for some (or equivalently, any) orthonormal basis $u_1, \ldots, u_m$ of W.

If equality holds in (1), W is said to be *calibrated by ω*.

The same if one substitutes $\mathbb{R}^n$ for a vector space V with an inner product.

Example

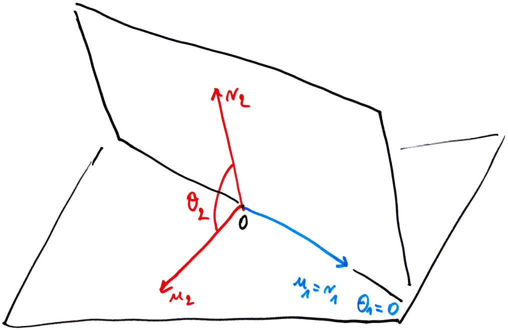
The m-form $e^{1\dots m}$ is an m-calibration on $\mathbb{R}^n$.

Idea of the proof. I will cheat with the orientation. Call $W_o = \operatorname{span}\{e_1, \dots, e_m\}$. Let W be any m-dimensional subspace of $\mathbb{R}^n$. We consider the angles between W and W_o.

The first, smaller, angle between W and W_o is θ_1, where

$$\cos\theta_1 = \text{máx}\left\{\langle u, v\rangle \mid u \in W_o, v \in W, |u| = |v| = 1\right\}.$$

(maximum inner product between unit vectors = minimum angle).



N_2
θ_2
O
μ_2
$\mu_1 = N_1$
$\theta_1 = 0$

Suppose $\cos\theta_1 = \langle u_1, v_1 \rangle$. Then the second angle is θ_2, where

$$\cos\theta_2 = \text{máx}\left\{ \langle u, v \rangle \mid u \in (u_1)^{\perp}, v \in (v_1)^{\perp}, |u| = |v| = 1 \right\}$$

(here $(u_1)^{\perp}$ and $(v_1)^{\perp}$ are the orthogonal complements of u_1 and v_1 in W_o and W, respectively).
Suppose $\cos\theta_2 = \langle u_2, v_2 \rangle$ and continue in this manner, obtaining orthonormal bases $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ of W_o and W, respectively.
One computes

$$\begin{aligned} \left| e^{1\ldots m}(v_1, \ldots, v_m) \right| &= \left| \cos\theta_1 \ldots \cos\theta_m \; e^{1\ldots m}(u_1, \ldots, u_m) \right| \\ &= \left| \cos\theta_1 \ldots \cos\theta_m \; e^{1\ldots m}(e_1, \ldots, e_m) \right| \leq 1. \end{aligned}$$

This also shows that span $\{e_1, \ldots, e_m\}$ is the unique m-dimensional subspace calibrated by $e^{1\ldots m}$. $\square$

Example

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{C}^n$. Then

$$\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\} =_{\text{notation}} \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$$

is a basis of $\mathbb{C}^n = \mathbb{R}^{2n}$ as an $\mathbb{R}$-vector space.

We consider the 2m-Wirtinger form $\omega_{m,n}$ on $\mathbb{R}^{2n}$, whose summands are associated to the complex m-dimensional coordinate subspaces. For instance, $\omega_{2,4}$ is the 4-form on $\mathbb{R}^8 = \mathbb{C}^4$ given by

$$e^{1526} + e^{1537} + e^{1548} + e^{2637} + e^{2648} + e^{3748}.$$

Recall that in this case the basis is

$$\{e_1, e_2, e_3, e_4, e_5 = ie_1, e_6 = ie_2, e_7 = ie_3, e_8 = ie_4\}.$$

Proposition

The $2m$-form $\omega_{m,n}$ is a $2m$-calibration on $\mathbb{R}^{2n}$ and the calibrated (real) $2m$-dimensional subspaces are exactly the complex m-dimensional subspaces of $\mathbb{C}^n = \mathbb{R}^{2n}$.

Idea of the proof. I will cheat with the orientation. Let W be any $2m$-dimensional subspace of $\mathbb{R}^{2n}$. There is a complex m-dimensional subspace W_o of $\mathbb{C}^n = \mathbb{R}^{2n}$ which is closest to W (in some sense). The angles between W and W_o turn out to be

$$0, \dots 0, \theta_1, \dots, \theta_m \qquad (m \text{ zeros})$$

and on a certain basis $v_1, \dots, v_{2m}$ of W one has

$$|\omega_{m,n}(v_1, \dots, v_{2m})| = |\cos\theta_1 \dots \cos\theta_m| \leq 1,$$

with equality only if $W = W_o$ (complex). $\square$

An important example of calibration is the so called special Lagrangian calibration on $\mathbb{R}^{2n}$. We won't deal with it in this course. Tomorrow more examples, related to the octonions.

Smooth Differential forms on manifolds

Let N be a smooth manifold. A smooth m-form Ω on N: At any point $p \in N$ we have an m-form Ω_p on T_pN depending smoothly on $p \in N$.
If Ω is a smooth m-form on N and M is a compact m-dimensional submanifold of N, then Ω can be integrated over M, that is,

$$\int_M \Omega$$

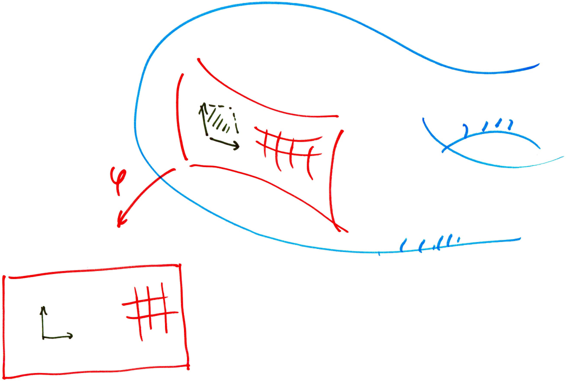
makes sense, as follows. Write M as an almost disjoint union of open sets A with compact closures $\overline{A}$ contained in coordinate patches $(U, \phi = (x_1, \ldots, x_m))$. Then

$$\int_{\overline{A}} \Omega = \int_{\phi(\overline{A})} \Omega\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right).$$

This is well defined by the theorem of change of variables, or more informally, because Ω_p is consistently a multiple of the m-volume at each $p \in M$.

$N = \mathbb{R}^3$

M surface $m = 2$



There is a differential operator d, called the exterior derivative, such that if Ω is a smooth m-form on N, then $d\Omega$ is a smooth $(m+1)$-form on N. I do not recall the definition.

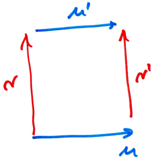
The smooth m-form Ω is said to be closed if $d\Omega = 0$.
For instance, if Ω is a constant m-form on $\mathbb{R}^n$, that is, $\Omega_p = \omega$ for all p, then $d\Omega = 0$.

Informal comment. The exterior derivative is in some sense the limit of an incremental quotient. Before taking the limit we have the following situation:

Case $m = 1$: The exterior derivative of a 1-form Ω is a 2-form $d\Omega$. The arguments of Ω are segments and those of Ω parallelograms. The value of $d\Omega$ on a parallelogram is an alternate sum of Ω applied to its sides.

Ω 1-form

$d\Omega$ 2-form

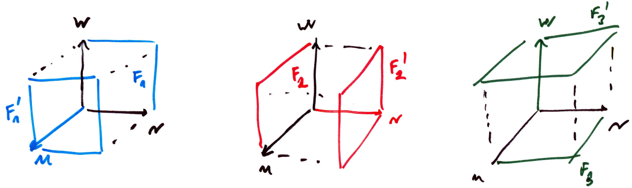


$d\Omega(u,v) \simeq$

$\Omega(v') - \Omega(v) + \Omega(u) - \Omega(u')$

Case $m = 2$**:** The exterior derivative of a 2-form Ω is a 3-form $d\Omega$. The arguments of Ω are parallelograms and those of $d\Omega$ parallelepipeds. The value of $d\Omega$ on a parallelepiped is an alternate sum of Ω applied to its faces.

Ω 2-form , $d\Omega$ 3-form



$$d\Omega(u,v,w) \approx \Omega(F_1') - \Omega(F_1)$$
$$+ \Omega(F_2) - \Omega(F_2')$$
$$+ \Omega(F_3') - \Omega(F_3)$$

As a consequence of Stoke's Theorem we have:

Proposition

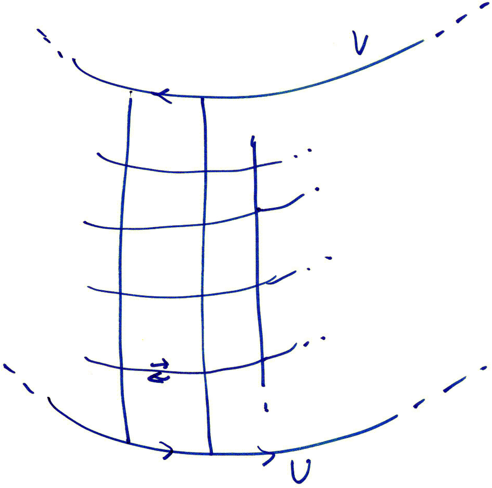
Let Ω be a closed smooth m-form on N and let U, V compact m-dimensional submanifolds of N with the same border which are homologous. Then

$$\int_U \Omega = \int_V \Omega.$$

Informal comment. If U and V are homologous, we think of them as the border of an $(m+1)$-dimensional manifold K. Write K as an almost disjoint union of cubes of dimension $m+1$.

Since $d\Omega = 0$ (Ω is closed), for each cube, the sum of the alternate values of Ω on m-dimensional faces is approximately zero.

Sum over all the cubes. The values of Ω on adjacent faces cancel and only the values of Ω on faces on the border remain.


V
U

Let N be an oriented Riemannian manifold, that is, at any point $p \in N$ we have an inner product $g_p = \langle ., . \rangle_p$ on $T_p N$ varying smoothly with p.

We have in particular the notion of length of curves in N and also the volume of open sets in N with compact closure, by integrating a smooth n-form Ω^g on N defined by

$$(\Omega^g)_p (u_1, \ldots, u_n) = 1$$

for $u_1, \ldots, u_n$ a positive orthonormal basis of $T_p N$.

Submanifolds of N inherit a Riemannian structures.

Definition

Let N be smooth Riemannian manifold. A smooth m-form Ω on N is said to be a smooth m-calibration on N if

a) Ω is closed (that is, $d\Omega = 0$) and

b) Ω_p is an m-calibration on T_pN for any $p \in N$

(since N is Riemannian, T_pN is an inner product vector space).

An m-dimensional submanifold M of N is said to be calibrated by Ω if $|\Omega_q(u_1, \ldots, u_m)| = 1$ for some (or equivalently any) orthonormal basis $u_1, \ldots, u_m$ of T_qM, for any $q \in M$.

Looking for calibrated submanifolds M is posing a PDE (constraints on the tangent spaces of M). The PDE may look awful in coordinates. If the problem has many symmetries, coordinates can be sometimes avoided.

Theorem

Fundamental Theorem of Calibrations. Let N be a Riemannian manifold and let Ω be a smooth m-calibration on N. Then m-dimensional closed manifolds calibrated by Ω are homologically volume minimizing in N. If $N = \mathbb{R}^n$, they are volume minimizing.

Proof. Let U be an open subset of M with compact closure and smooth border ∂U (possibly $\partial U = \emptyset$) and let V be a compact submanifold V (same dimension) homologous to U in N with $\partial V = \partial U$. We compute

$$\mathrm{vol}\,(U) \overset{(1)}{=} \int_U \Omega^g \overset{(2)}{=} \int_U \Omega \overset{(3)}{=} \int_V \Omega \overset{(4)}{\leq} \int_V \Omega^g \overset{(5)}{=} \mathrm{vol}\,(V)\,.$$

Equalities (1) and (5) are just the definition of the volume of a compact submanifold of a Riemannian manifold (N, g). Equality (3) follows from the corollary of Stokes Theorem we saw above. Equality (2) and inequality (4) follow from the facts that Ω is a smooth calibration and M is calibrated by Ω. $\square$

Example

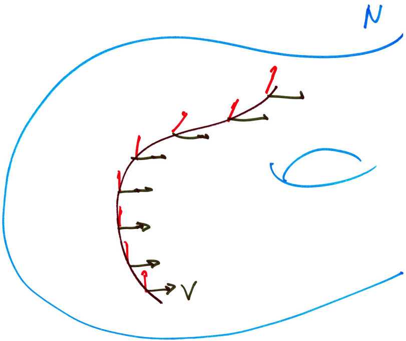
$\mathbb{R}^m \times \{0\}$ has minimum volume in $\mathbb{R}^n$. Use the constant calibration $\Omega \equiv e^{1\dots m}$.

Example

Any closed m-dimensional complex submanifold M of $N = \mathbb{C}^n$ is homologically volume minimizing. Use the constant $2m$-Wirtinger calibration $\Omega \equiv \omega_{m,n}$.

More generally, any closed complex submanifold of a Kähler manifold is homologically volume minimizing.

We recall that a Kähler manifold N is a Riemannian manifold such that at any $p \in N$ we have a rotation J_p through the angle $\pi/2$ on T_pN (that is, J is orthogonal with $J_p^2 = -\,\mathrm{id}$) in such a way that if V is parallel vector field along a curve γ in N, then $J(V)$ is also parallel along γ.



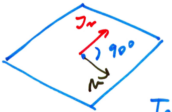
N Kähler

γ curve in N

V parallel tangent vector field along γ

$\Longrightarrow$

JV parallel along γ



T_pN

Calibrations related to the octonions

Definition

A normed division algebra $(\mathbb{A}, +, \cdot, \langle , \rangle)$ is a finite dimensional algebra $(\mathbb{A}, +, \cdot)$ with an inner product such that

a) $\|xy\| = \|x\| \, \|y\|$ for any $x, y \in \mathbb{A}$,

b) any $x \neq 0$ has an inverse.

There are only four of them: $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

They have dimensions 1,2,4 and 8, respectively.

$\mathbb{C}$ is not ordered, $\mathbb{H}$ is not commutative and $\mathbb{O}$ is not even associative.

Each one can be constructed from the preceding:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i, \qquad \mathbb{H} = \mathbb{C} + \mathbb{C}j.$$

Writing $z, w \in \mathbb{C}$ as $z = a + bi$, $w = c + di$, we have

$$\begin{aligned} \mathbb{H} \ni q &= z + wj = a + bi + (c + di)\, j \\ &= a + bi + cj + dk = a + \mathrm{Im}(q), \end{aligned}$$

where i, j, k are as usual anticommuting square roots of -1.

The octonions are $\mathbb{O} = \mathbb{H} + \mathbb{H}e$ with $e^2 = -1$.
They have 7 independent anticommuting square roots of -1.
The general octonion has the form

$$x = a + bi + cj + dk + \alpha e + \beta ie + \gamma je + \delta ke = a + \mathrm{Im}(x)$$

One can write a table with the values of the multiplication of the generators, for instance

$$j\,(ie) = ke.$$

Lack of associativity: $j\,(ie) = ke$ is not equal to $(ji)\,e = -ke$.

The associative calibration.

Consider the 3-form ϕ on $\operatorname{Im}\mathbb{O} = \mathbb{R}^7$ defined by

$$\phi(x, y, z) = \langle x, yz \rangle$$

(it is in fact alternating). Putting

$$(i, j, k, e, ie, je, ke) = (e_1, e_2, e_3, e_4, e_5, e_6, e_7)$$

(the canonical basis of $\mathbb{R}^7$), then

$$\phi = e^{123} + e^{145} + e^{176} + e^{246} + e^{257} + e^{347} + e^{365}.$$

The first term e^{123} corresponds to the fact that $ij = k$, or equivalently, $e_1 e_2 = e_3$.
The fifth term e^{257} corresponds to the fact that $j(ie) = ke$, or equivalently, $e_2 e_5 = e_7$.

There are many ways of injecting $\mathbb{C}$ into $\mathbb{H}$:

Take any $u \in \operatorname{Im} \mathbb{H}$ with $\|u\| = 1$, then $u^2 = -1$ and this implies that $\{a + bu \mid a, b \in \mathbb{R}\}$ is a copy of $\mathbb{C}$ in $\mathbb{H}$.

There is an S^2 worth of injections of $\mathbb{C}$ into $\mathbb{H}$.

In the same way, there are many (an 8 dimensional manifold, $G_2/S^3 \times S^3$) subspaces of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

They are called the associative subspaces of $\mathbb{O}$.

Proposition

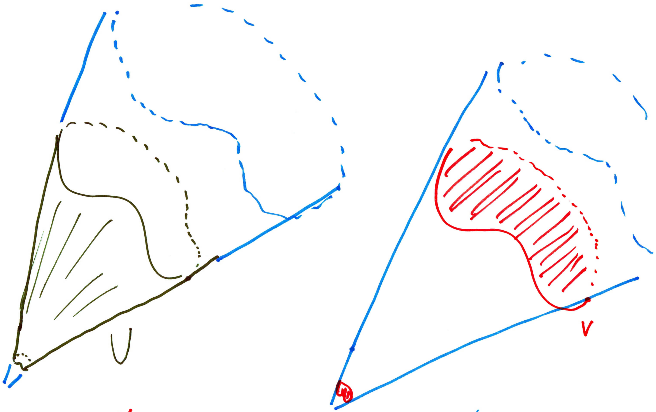
ϕ is a 3-calibration on $\operatorname{Im}\mathbb{O} = \mathbb{R}^7$ and the calibrated subspaces are exactly the imaginary parts of the associative subspaces of $\mathbb{O}$.

Proof. If follows from the following algebraic identity:

$$\phi(x, y, z)^2 + \frac{1}{4}\|(xy)z - x(yz)\|^2 = 1$$

holds for any orthonormal set $\{x, y, z\}$ in $\operatorname{Im}\mathbb{O}$. $\square$

$M = \mathbb{R}^3$ $\quad$ M cone, $\dim M = 2$



$\partial U = \partial V$ $\quad$ $\mathrm{area}(V) < \mathrm{area}(U)$

Volume minimizing cones

In higher dimensions and codimensions the singularity at the origin of some cones does not prevent them from being volume minimizing.

In fact, we will give examples of volume minimizing 3-dimensional cones in $\mathbb{R}^7$ with a singularity at the origin.

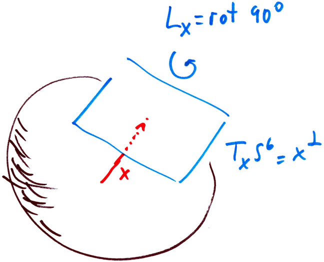
Before, we need the almost complex structure on S^6, the sphere of dimension 6 in $\mathbb{R}^7 = \operatorname{Im} \mathbb{H}$.

At any point $x \in S^6$ consider the linear operator

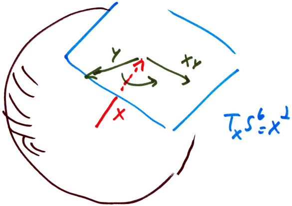
$$L_x : T_x S^6 = x^\perp \to T_x S^6 = x^\perp, \qquad L_x(y) = xy.$$

It is orthogonal and satisfies $(L_x)^2 = -\operatorname{id}$ (a rotation through angle $\pi/2$ on each tangent space $T_x S^6$).

Comment: $\left(S^6, L\right)$ is not Kähler.



$S^6 \subset \mathbb{R}^7 = \mathrm{Im}\,\mathbb{H}$



S^6

Given a compact surface M in S^6, the cone M in $\mathbb{R}^7$ generated by K is defined by

$$M = \{ty \mid y \in K, t > 0\}.$$

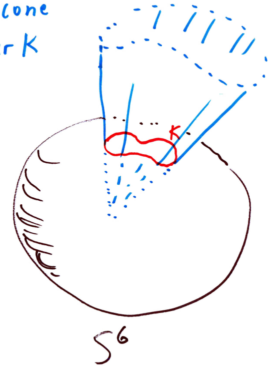
Proposition

Let K be a complex surface in S^6 (that is, $T_yK \subset T_yS^6$ is invariant for L_y for each $y \in K$). Then the cone M in $\mathbb{R}^7$ generated by K is volume minimizing.

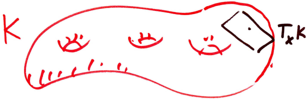
Proof. The cone M is calibrated by the associative calibration ϕ. This follows from the fact that a plane W in $x^\perp = T_xS^6$ is invariant by L_x if and only if span $\{1, x, W\}$ is associative. $\square$

Remark. Robert Bryant proved that any compact Riemann surface (in particular, of any genus) can be taken as K above.

M cone over K



dim K = 2



topologically

$$L_x(T_x K) = T_x K$$

Next, we give an example of a function $F : \mathbb{R}^4 \to \mathbb{R}^3$ which is not of class C^1 but whose graph $\subset \mathbb{R}^7$ is volume minimizing.

The coassociative calibration

Consider the 4-form $\psi = *\phi$ in $\mathbb{R}^7$, that is

$$\psi = e^{4567} + e^{2367} + e^{2345} + e^{1357} + e^{1346} + e^{1256} + e^{1274}.$$

(compare with $\phi = e^{123} + e^{145} + e^{176} + e^{246} + e^{257} + e^{347} + e^{365}$).

Proposition

ψ is a 4-calibration of $\mathbb{R}^7$ and the calibrated subspaces are exactly the orthogonal complements of the associative subspaces.

Proposition

The function $F : \mathbb{H} = \mathbb{R}^4 \to \operatorname{Im} \mathbb{H} = \mathbb{R}^3$ *defined by*

$$F(q) = \frac{\sqrt{5}}{2} \frac{q i \bar{q}}{\|q\|} \quad \text{for } q \neq 0, \qquad F(0) = 0$$

is Lipschitz continuous but not of class C^1*.*

The tangent spaces to its graph are calibrated by ψ *anf hence graph* (F) *is volume minimizing.*

Remark. A classical result says that if the graph of a C^1 function is volume minimizing, then the function is real analytic.

Comment. The function $f : S^3 \subset \mathbb{H} \to S^2 \subset \operatorname{Im} \mathbb{H}$, defined by

$$f(q) = qi\overline{q}$$

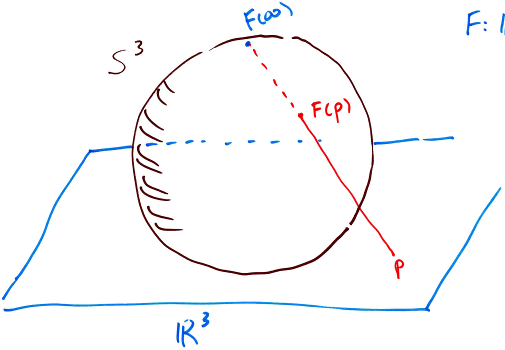
(the restriction of F to S^3, except for the constant $\sqrt{5}/2$) is the Hopf map, whose level sets are the great circles in S^3 obtained as the intersection of S^3 with complex lines (one dimensional complex subspaces) in $\mathbb{C}^2 = \mathbb{H}$.

Visualization of the Hopf map

$F : \mathbb{R}^3 \cup \{\infty\} \to S^3$ stereographic map.

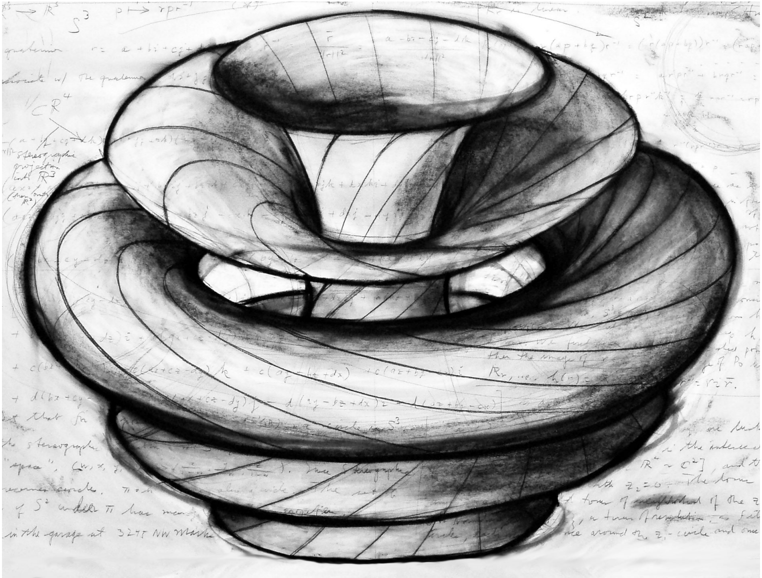
$$F\,(z\text{-axis} \cup \{\infty\}) = \text{great circle } \ (\{0\} \times \mathbb{C}) \cap S^3 \subset \mathbb{C}^2$$

The complement of the z-axis in $\mathbb{R}^3$ can be written as the smooth disjoint union of those circles C such that $F\,(C)$ are the remaining great circles in S^3 obtained by intersecting S^3 with complex lines.



$F: \mathbb{R}^3 \cup \{\infty\} \to S^3$

stereographic map



S^3
Stereographic projection onto $\mathbb{R}^3$

Invariant calibrations on homogeneous spaces

Let G be a Lie group acting transitively by isometries on a Riemannian manifold N.

Consider a calibration ω on T_pN for some $p \in N$.

If you are lucky, ω is invariant by the isotropy subgroup at p and then you have a smooth form Ω on N.

If you are lucky again, Ω is closed and you have a smooth calibration on N.

Fact: Invariance and closedness are unrelated notions.

The best unit vector field on S^3

Let M be a Riemannian manifold. The Sasaki metric on its unit tangent bundle T^1M is the Riemannian metric such that for any smooth curve $v : [a, b] \to T^1M$

$$\text{length}(v) = \int_a^b \sqrt{\|\alpha'(t)\|^2 + \|v'(t)\|^2}\, dt,$$

where α is the foot point curve of v and v' means the covariant derivative of v along α ($\|v'(t)\|$ measures to which extent v fails to be parallel along α).

A geodesic in T^1M does not necessarily project to a geodesic in M.

The unit tangent bundle T^1S^3 is diffeomorphic to $S^3 \times S^2$, since S^3 is parallelizable.
The Sasaki metric on T^1S^3 is not the product metric.

Theorem

(Gluck and Ziller, 1983) The image of the Hopf vector field on S^3 is homologically volume minimizing in T^1S^3.

Proof. They use a smooth calibration on T^1S^3 invariant by the action of the group $S^3 \times S^3$, which acts transitively on T^1S^3. $\square$

A 3-calibration on $\mathbb{R}^6$ with exactly two calibrated subspaces

In 1983 (one year after the publication of the pioneering paper on calibrations by Harvey and Lawson), Dadok and Harvey and independently, the same year, Frank Morgan, studied calibrations on $\mathbb{R}^6$ and found new ones, among them one the form

$$\omega = e^{123} + \lambda \left(e^{156} + e^{426} + e^{453} \right) + \mu e^{456},$$

for some $\lambda, \mu \in \mathbb{R}$, which has exactly two calibrated 3-dimensional subspaces.

This is an interesting result in multilinear algebra, but since the calibrated subspaces are not allowed to move in a continuum, ω is not useful to prove that a three dimensional submanifold of $\mathbb{R}^6$ is volume minimizing (except, of course, for affine subspaces).

In 2001 I considered the Lie group $G = S^3 \times S^3$ with an invariant Riemannian metric (not the product metric) and the invariant smooth calibration Ω on G such that $\Omega_e = \omega$ above (previous some identification of $\mathbb{R}^6$ with $T_e G$).

The form Ω turned out to be closed. I also found the corresponding family of calibrated submanifolds, two of them in distinct homology classes.

Calibrations in pseudo-Riemannian Geometry

A pseudo-Riemannian manifold (N, g) is the same as a Riemannian manifold, but the inner product g_p on T_pN is not necessarily positive definite, it may have signature.

For instance $N = \mathbb{R}^{1,1}$, that is, $\mathbb{R}^2$ with the inner product of signature $(1,1)$ given by

$$\langle (x,y), (x,y) \rangle = x^2 - y^2.$$

On a pseudo-Riemannian manifold, from each point we have directions of two different natures: time-like and space-like (and also the borderline null directions).

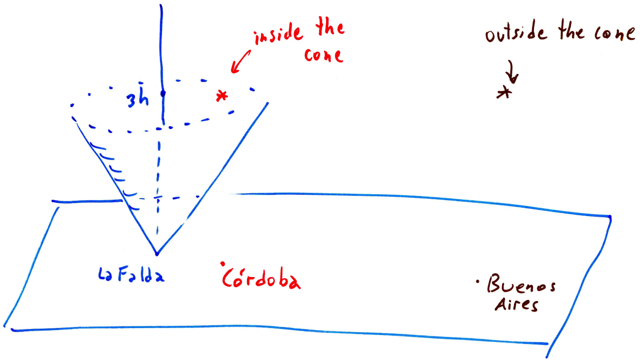
Example

Space-time $\mathbb{R}^{3,1}$ (the fact that nothing travels faster than light).

A humbler analogue: Plane-time $\mathbb{R}^{2,1}$.

Suppose a bus travels with velocity 100 km/h.
With that bus, I can travel from La Falda to Córdoba in less than 3 h, but I cannot arrive to Buenos Aires in 3 h.

Put the origin at La Falda and a convenient inner product of signature $(2, 1)$ on $\mathbb{R}^2 \times \mathbb{R}$.
The pair (Córdoba , 3 h) $\in \mathbb{R}^2 \times \mathbb{R}$ is time-like and the pair (Buenos Aires , 5 h) is space-like.


inside the cone
outside the cone
3h
La Falda
Córdoba
Buenos Aires

Example

The manifold $\mathcal{L}$ of all (unparametrized) straight lines of $\mathbb{R}^3$ has dimension four.

Generic curves in $\mathcal{L}$ are of two types, those describing right handed or left handed screws.

This induces on $\mathcal{L}$ a pseudo-Riemannian metric of signature $(2,2)$ invariant by the group of rigid transformations of $\mathbb{R}^3$.

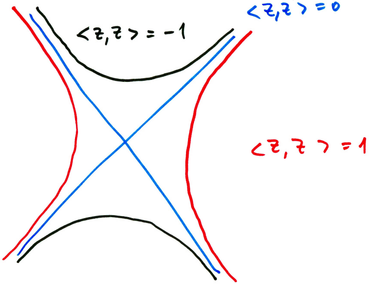
Y. Godoy, M. Salvai, *Calibrated geodesic foliations of hyperbolic space*, Proc. Amer. Math. Soc., 2016.

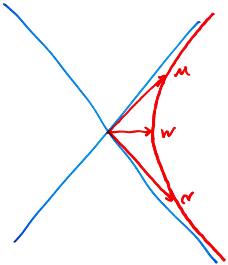
Fact. For pseudo-Riemannian manifolds calibrations are defined in a slightly different manner. They are useful to prove that submanifolds maximize volume.

Example

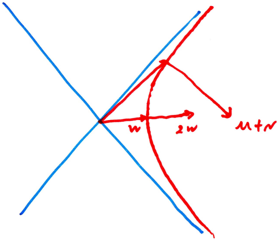
Two space-like vectors $u, v \in \mathbb{R}^{1,1}$ such that $\|u + v\| > \|u\| + \|v\|$.

$\mathbb{R}^{1,1}$





u, v, w unit vectors



$\|u\| + \|v\| = 1 + 1 = 2$

$\|u+v\| > \|2w\| = 2\|w\| = 2 \cdot 1 = 2$

Example

Optimal transport maps are calibrated. Given a compact Riemannian manifold K with two measures μ and ν with the same total volume, the graph of the optimal map transporting μ to ν is a calibrated submanifold of $K \times K$ for some natural pseudo-Riemannian metric of signature (k, k) on the product.

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