usc (lsc, continuous) at every limit point of $E$ which is in $E$. The next theorem characterizes functions which are semicontinuous relative to a set.

**(4.14) Theorem**

(i) *A function $f$ is usc relative to $E$ if and only if $\{\mathbf{x} \in E : f(\mathbf{x}) \geq a\}$ is relatively closed [equivalently, $\{\mathbf{x} \in E : f(\mathbf{x}) < a\}$ is relatively open] for all finite $a$.*

(ii) *A function $f$ is lsc relative to $E$ if and only if $\{\mathbf{x} \in E : f(\mathbf{x}) \leq a\}$ is relatively closed [equivalently, $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$ is relatively open] for all finite $a$.*

*Proof.* Statements (i) and (ii) are equivalent since $f$ is usc if and only if $-f$ is lsc. It is therefore enough to prove (i). Suppose first that $f$ is usc relative to $E$. Given $a$, let $\mathbf{x}_0$ be a limit point of $\{\mathbf{x} \in E : f(\mathbf{x}) \geq a\}$ which is in $E$. Then there exist $\mathbf{x}_k \in E$ such that $\mathbf{x}_k \to \mathbf{x}_0$ and $f(\mathbf{x}_k) \geq a$. Since $f$ is usc at $\mathbf{x}_0$, we have $f(\mathbf{x}_0) \geq \limsup_{k\to\infty} f(\mathbf{x}_k)$. Therefore, $f(\mathbf{x}_0) \geq a$, so that $\mathbf{x}_0 \in \{\mathbf{x} \in E : f(\mathbf{x}) \geq a\}$. This shows that $\{\mathbf{x} \in E : f(\mathbf{x}) \geq a\}$ is relatively closed.

Conversely, let $\mathbf{x}_0$ be a limit point of $E$ which is in $E$. If $f$ is not usc at $\mathbf{x}_0$, then $f(\mathbf{x}_0) < +\infty$, and there exist $M$ and $\{\mathbf{x}_k\}$ such that $f(\mathbf{x}_0) < M$, $\mathbf{x}_k \in E$, $\mathbf{x}_k \to \mathbf{x}_0$, and $f(\mathbf{x}_k) \geq M$. Hence, $\{\mathbf{x} \in E : f(\mathbf{x}) \geq M\}$ is not relatively closed since it does not contain all its limit points which are in $E$.

**(4.15) Corollary** *A finite function $f$ is continuous relative to $E$ if and only if all sets of the form $\{\mathbf{x} \in E : f(\mathbf{x}) \geq a\}$ and $\{\mathbf{x} \in E : f(\mathbf{x}) \leq a\}$ are relatively closed [or, equivalently, all $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$ and $\{\mathbf{x} \in E : f(\mathbf{x}) < a\}$ are relatively open] for finite $a$.*

**(4.16) Corollary** *Let $E$ be measurable, and let $f$ be defined on $E$. If $f$ is usc (lsc, continuous) relative to $E$, then $f$ is measurable.*

*Proof.* Let $f$ be usc relative to $E$. Since $\{\mathbf{x} \in E : f(\mathbf{x}) \geq a\}$ is relatively closed, it is the intersection of $E$ with a closed set. Hence, it is measurable, and the result follows from (4.1).

The results in (4.14)–(4.16) deserve special attention in certain cases. Suppose, for example, that $E = \mathbf{R}^n$ and $f$ is usc everywhere in $\mathbf{R}^n$. Since $\{f > a\} = \bigcup_{k=1}^{\infty} \{f \geq a + 1/k\}$, it follows from (4.14) that $\{f > a\}$ is of type $F_\sigma$. Since an $F_\sigma$ set is a Borel set, we see that a function which is usc (similarly, lsc or continuous) at every point of $\mathbf{R}^n$ is Borel measurable.

## 3. Properties of Measurable Functions: Egorov's Theorem and Lusin's Theorem

Our next theorem states in effect that if a sequence of measurable functions converges at each point of a set $E$, then, with the exception of a subset of $E$ with arbitrarily small measure, the sequence actually converges *uniformly.*



This remarkable result cannot hold, at least in the form just stated, without some further restrictions. For example, if $E = \mathbf{R}^n$ and $f_k = \chi_{\{\mathbf{x}:|\mathbf{x}|<k\}}$, then $f_k$ converges to 1 everywhere but does not converge uniformly outside any bounded set. Again, if the $f_k$ are finite but the limit $f$ is infinite in a set of positive measure, then $|f_k - f|$ is also infinite in this set. The difficulties in these examples can be easily overcome: the missing ingredient in the first case is that $|E| < +\infty$, and in the second, that $|f| < +\infty$ a.e. Adding these restrictions, we obtain the following basic result.

**(4.17) Theorem (*Egorov's Theorem*)** *Suppose that $\{f_k\}$ is a sequence of measurable functions which converges almost everywhere in a set $E$ of finite measure to a finite limit $f$. Then given $\varepsilon > 0$, there is a closed subset $F$ of $E$ such that $|E - F| < \varepsilon$ and $\{f_k\}$ converges uniformly to $f$ on $F$.*

In order to prove this, we need a preliminary result which is interesting in its own right.

**(4.18) Lemma** *Under the same hypothesis as in Egorov's theorem, given $\varepsilon, \eta > 0$, there is a closed subset $F$ of $E$ and an integer $K$ such that $|E - F| < \eta$ and $|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon$ for $\mathbf{x} \in F$ and $k > K$.*

*Proof.* Fix $\varepsilon, \eta > 0$. For each $m$, let $E_m = \{|f - f_k| < \varepsilon$ for all $k > m\}$. Thus, $E_m = \bigcap_{k>m} \{|f - f_k| < \varepsilon\}$, so that $E_m$ is measurable. Clearly, $E_m \subset E_{m+1}$. Moreover, since $f_k \to f$ a.e. in $E$ and $f$ is finite, $E_m \nearrow E - Z$, $|Z| = 0$. Hence, by (3.26), $|E_m| \to |E - Z| = |E|$. Since $|E| < +\infty$, it follows that $|E - E_m| \to 0$. Choose $m_0$ so that $|E - E_{m_0}| < \frac{1}{2}\eta$, and let $F$ be a closed subset of $E_{m_0}$ with $|E_{m_0} - F| < \frac{1}{2}\eta$. Then $|E - F| < \eta$ and $|f - f_k| < \varepsilon$ in $F$ if $k > m_0$.

*Proof of Egorov's theorem.* Given $\varepsilon > 0$, use (4.18) to select closed $F_m \subset E$, $m \geq 1$, and integers $K_{m,\varepsilon}$ such that $|E - F_m| < \varepsilon 2^{-m}$ and $|f - f_k| < 1/m$ in $F_m$ if $k > K_{m,\varepsilon}$. The set $F = \bigcap_m F_m$ is closed, and since $F \subset F_m$ for all $m$, $f_k$ converges uniformly to $f$ on $F$. Finally, $E - F = E - \bigcap F_m = \bigcup (E - F_m)$ and, therefore, $|E - F| \leq \sum |E - F_m| < \varepsilon$. This completes the proof.

See Exercises 13 and 14 for an analogue of Egorov's theorem in the continuous parameter case; i.e., in the case when $f_{\mathbf{y}}(\mathbf{x}) \to f(\mathbf{x})$ as $\mathbf{y} \to \mathbf{y}_0$.

We have observed that a continuous function is measurable. Our next result, Lusin's theorem, gives a continuity property which characterizes measurable functions. In order to state the result, we first make the following definition. A function $f$ defined on a measurable set $E$ has *property $\mathscr{C}$* on $E$ if given $\varepsilon > 0$, there is a closed set $F \subset E$ such that

(i) $|E - F| < \varepsilon$,