(ii) f is continuous relative to F.

We recall that condition (ii) means that if $\mathbf{x}_0$ and $\{\mathbf{x}_k\}$ belong to F and $\mathbf{x}_k \to \mathbf{x}_0$, then $f(\mathbf{x}_k) \to f(\mathbf{x}_0)$. In case F is bounded (and, therefore, compact), (ii) implies that the restriction of f to F is uniformly continuous [theorem (1.15)].

(4.19) Lemma *A simple measurable function has property $\mathscr{C}$.*

Proof. Suppose that f is a simple measurable function on E, taking distinct values $a_1, \dots, a_N$ on measurable subsets $E_1, \dots, E_N$. Given $\varepsilon > 0$, choose closed $F_j \subset E_j$ with $|E_j - F_j| < \varepsilon/N$. Then the set $F = \bigcup_{j=1}^N F_j$ is closed, and since $E - F = \bigcup E_j - \bigcup F_j \subset \bigcup (E_j - F_j)$, we have $|E - F| \le \sum |E_j - F_j| < \varepsilon$. It remains only to show that f is continuous on F. Note that each F_j is relatively open in F, so the only points of F in a small neighborhood of any point of F_j are points of F_j itself. The continuity on F of f follows from this since f is constant on each F_j.

Property $\mathscr{C}$ is actually equivalent to measurability, as we now show.

(4.20) Theorem (*Lusin's Theorem*) *Let f be defined and finite on a measurable set E. Then f is measurable if and only if it has property $\mathscr{C}$ on E.*

Proof. If f is measurable, then by (4.13) there exist simple measurable f_k, $k = 1, 2, \dots$, which converge to f. By (4.19), each f_k has property $\mathscr{C}$, so given $\varepsilon > 0$, there exist closed $F_k \subset E$ such that $|E - F_k| < \varepsilon 2^{-k-1}$ and f_k is continuous relative to F_k. Assuming for the moment that $|E| < +\infty$, we see by Egorov's theorem that there is a closed $F_0 \subset E$ with $|E - F_0| < \frac{1}{2}\varepsilon$ such that $\{f_k\}$ converges uniformly to f on F_0. If $F = F_0 \cap (\bigcap_{k=1}^\infty F_k)$, then F is closed, each f_k is continuous relative to F, and $\{f_k\}$ converges uniformly to f on F. Hence, f is continuous relative to F by (1.16). Since

$$|E - F| \le |E - F_0| + \sum_{k=1}^{\infty} |E - F_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

it follows that f has property $\mathscr{C}$ on E. This proves the necessity of property $\mathscr{C}$ for measurability if $|E| < +\infty$.

If $|E| = +\infty$, write $E = \bigcup_{k=1}^\infty E_k$, where E_k is the part of E in the ring $\{\mathbf{x} : k - 1 \le |\mathbf{x}| < k\}$. Since $|E_k| < +\infty$, we may select closed $F_k \subset E_k$ such that $|E_k - F_k| < \varepsilon 2^{-k}$ and f is continuous relative to F_k. If $F = \bigcup_{k=1}^\infty F_k$, it follows that $|E - F| \le \sum |E_k - F_k| < \varepsilon$, and that f is continuous relative to F. A simple argument shows that F is closed, and therefore f has property $\mathscr{C}$ on E.

Conversely, suppose that f has property $\mathscr{C}$ on E. For each k, $k = 1, 2, \dots$, choose a closed $F_k \subset E$ such that $|E - F_k| < 1/k$ and the restriction of f to

F_k is continuous. If $H = \bigcup_{k=1}^\infty F_k$, then $H \subset E$ and the set $Z = E - H$ has measure zero. We have

$$\{\mathbf{x} \in E : f(\mathbf{x}) > a\} = \{\mathbf{x} \in H : f(\mathbf{x}) > a\} \cup \{\mathbf{x} \in Z : f(\mathbf{x}) > a\}$$
$$= \bigcup_{k=1}^{\infty} \{\mathbf{x} \in F_k : f(\mathbf{x}) > a\} \cup \{\mathbf{x} \in Z : f(\mathbf{x}) > a\}.$$

Since $\{\mathbf{x} \in Z : f(\mathbf{x}) > a\}$ has measure zero, the measurability of f will follow from that of each $\{\mathbf{x} \in F_k : f(\mathbf{x}) > a\}$. However, since f is continuous relative to F_k and F_k is measurable, $\{\mathbf{x} \in F_k : f(\mathbf{x}) > a\}$ is measurable by (4.16). This completes the proof of Lusin's theorem.

4. Convergence in Measure

Let f and $\{f_k\}$ be measurable functions which are defined and finite a.e. in a set E. Then $\{f_k\}$ is said to *converge in measure* on E to f if for every $\varepsilon > 0$,

$$\lim_{k\to\infty} |\{\mathbf{x} \in E : |f(\mathbf{x}) - f_k(\mathbf{x})| > \varepsilon\}| = 0.$$

We will indicate convergence in measure by writing

$$f_k \xrightarrow{m} f.$$

This concept has many useful applications in analysis. Here we will discuss its relation to ordinary pointwise convergence; the first result is basically a reformulation of (4.18).

(4.21) Theorem *Let f and f_k, $k = 1, 2, \dots$, be measurable and finite a.e. in E. If $f_k \to f$ a.e. on E and $|E| < +\infty$, then $f_k \xrightarrow{m} f$ on E.*

Proof. Given $\varepsilon, \eta > 0$, let F and K be as defined in lemma (4.18). Then if $k > K$, $\{\mathbf{x} \in E : |f(\mathbf{x}) - f_k(\mathbf{x})| > \varepsilon\} \subset E - F$, and since $|E - F| < \eta$, the result follows.

We recall that the conclusion above may not hold if $|E| = +\infty$, as shown by the example $E = \mathbf{R}^n$, $f_k = \chi_{\{\mathbf{x}:|\mathbf{x}|<k\}}$, and $f = 1$.

Convergence in measure does not imply pointwise convergence a.e., even for sets of finite measure. To see this, take $n = 1$ and let $\{I_k\}$ be a sequence of subintervals of [0,1] satisfying the following conditions:

(i) Each point of [0,1] belongs to infinitely many I_k.
(ii) $\lim_{k\to\infty} |I_k| = 0$.

For example, let the first interval be [0,1], the next two be the two halves of [0,1], the next four be the four quarters, and so on. Then if $f_k = \chi_{I_k}$, we have $f_k \xrightarrow{m} 0$, while f_k diverges at every point of [0,1].

There is, however, the following partial converse to (4.21).