WEIGHTS FOR COMMUTATORS OF THE ONE-SIDED DISCRETE SQUARE FUNCTION, THE WEYL FRACTIONAL INTEGRAL AND OTHER ONE-SIDED OPERATORS.

M. Lorente and M.S. Riveros

Abstract. The purpose of this paper is to prove strong type inequalities with one-sided weights for commutators (with symbol $b \in BMO$) of several one-sided operators, such as the one-sided discrete square function, the one-sided fractional operators, or one-sided maximal operators given by the convolution with a smooth function. We also prove that $b \in BMO$ is a necessary condition for the boundedness of commutators of these one-sided operators.

1. Introduction

There is a great amount of works that deal with the topic of commutators of different operators with BMO functions. If T is the operator given by convolution with a kernel K, the commutator of T, with a locally integrable function b, called the symbol, is the operator

$$T_b f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)f(y)dy.$$

Sometimes the symbol appears inside the integral with absolute values (see Section 2 for precise definitions).

In 1976, Coifman, Rochberg and Weiss, [4], introduced the higher order commutators,

$$T_b^k f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k K(x-y)f(y)dy, \quad k = 0, 1, 2, ...$$

Nottice that $T_b^0 = T$. In [4], it is proved that if K is a Calderón-Zygmund kernel then T_b^k is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ when $b \in BMO$. In fact, the condition $b \in BMO$ is also necessary in order to have T_b bounded on $L^p(\mathbb{R}^n)$.

Later, many authors have studied strong and weak type inequalities for commutators with weights (see [3], [13], [14]). Furthermore, many of the

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results have been generalized to commutators of other operators, not only Calderón-Zygmund operators (see [16], [17], [18]).

In this paper we are interested in studying commutators of certain one-sided operators, such as the one-sided discrete square function that appears in [19], the fractional integrals of Weyl and Riemann-Liouville and a class of one-sided maximal operators given by the convolution with a smooth function. This work is highly inspired in the works of C. Segovia and J. L. Torrea [17] and [18]. The main tools for proving the weighted inequalities for these one-sided operators are extrapolation theorems proved by R. Macías and M.S. Riveros in [6]. We also prove that the condition $b \in BMO$ is necessary, i.e., even though we have one-sided operators and our weights are one-sided A_p weights, the condition $b \in BMO$ can't be weakened (is a two-sided condition).

Throughout this paper the letter C will be a positive constant, not necessarily the same at each occurrence. If $1 \leq p \leq \infty$, then its conjugate exponent will be denoted by p' and A_p will be the classical Muckenhoupt's class of weights (see [12]).

2. Definitions and statement of the results

Definition 2.1. For f locally integrable, we define the one-sided discrete square function applied to f by

$$S^+f(x) = \left(\sum_{n\in\mathbb{Z}} |A_n f(x) - A_{n-1}f(x)|^2 \right)^{1/2},$$

where $A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y)dy$.

It is not difficult to see that $S^+f(x) = ||U^+f(x)||_{\ell^2}$, where U^+ is the sequence valued operator

$$U^+f(x) = \int_{\mathbb{R}} H(x-y)f(y)dy, \tag{2.1}$$

where

$$H(x) = \left\{ \frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x) \right\}_{n\in\mathbb{Z}}. \tag{2.2}$$

(See [19].)

Definition 2.2. The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f| \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f|.$$

The good weights for these operators are the one-sided weights, A_p^+ and A_p^-:

$$\sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b \omega \left(\int_b^c \omega^{1-p'} \right)^{p-1} < \infty, \quad 1<p<\infty, \tag{A_p^+}$$

$$(A_1^+) \qquad M^-\omega(x) \le C\omega(x) \quad \text{a.e.}$$

and

$$(A_\infty^+) \qquad A_\infty^+ = \cup_{p\ge 1} A_p^+.$$

The classes A_p^- are defined in a similar way. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneq A_p^+$ and $A_p \subsetneq A_p^-$. (See [15], [7], [8], [9] for more definitions and results.)

It is proved in [19], that $\omega \in A_p^+$, $1 < p < \infty$, if, and only if, S^+ is bounded from $L^p(\omega)$ to $L^p(\omega)$ and that $\omega \in A_1^+$, if, and only if, S^+ is of weak-type (1,1) with respect to ω.

We shall also use for our purposes the following variant of the one-sided Hardy-Littlewood maximal operator:

Definition 2.3. Let $\varphi \in \mathcal{C}_c^\infty(-\infty, 0]$, $\varphi \ge 0$ nondecreasing in $(-\infty, 0]$. For $\varepsilon > 0$ let $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$. The maximal operator associated to φ is defined by

$$M_\varphi^+ f(x) = \sup_{\varepsilon>0} \varphi_\varepsilon * |f|(x).$$

It is not difficult to see that $M_\varphi^+ f$ is pointwise equivalent to $M^+ f$. As a consequence, M_φ^+ is bounded from $L^p(\omega)$ to $L^p(\omega)$, for $1 < p < \infty$ and $\omega \in A_p^+$.

Definition 2.4. The one-sided maximal fractional operator M_α^+, $0 < \alpha < 1$, is defined, for locally integrable functions f, by

$$M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f|.$$

It is proved in [2] that $||(M_\alpha^+ f)\omega||_q \le C||f\omega||_p$ if and only if $\omega \in A^+(p,q)$, for $1 < p < q$, $1/p - 1/q = \alpha$, where

$$(A^+(p,q)) \qquad \left(\frac{1}{h}\int_{x-h}^x \omega^q\right)^{1/q} \left(\frac{1}{h}\int_x^{x+h} \omega^{-p'}\right)^{1/p'} \le C,$$

$$(A^+(p,\infty)) \qquad ||\omega\chi_{[x-h,x]}||_\infty \left(\frac{1}{h}\int_x^{x+h} \omega^{-p'}\right)^{1/p'} \le C,$$

for all $h > 0$ and $x \in \mathbb{R}$.

We also have a variant of the operator M_α^+:

Definition 2.5. Let $0 < \alpha < 1$ and let $\varphi_\alpha \in \mathcal{C}^\infty((-\infty, 0])$, $\varphi_\alpha \ge 0$, nondecreasing in $(-\infty, 0]$ and such that $|\varphi_\alpha(x-y) - \varphi_\alpha(x)| \le C|y||x|^{-2+\alpha}$, for all x, y such that $|x| > 2|y|$. For each $\varepsilon > 0$, set $\varphi_{\alpha,\varepsilon}(x) = \varepsilon^{-1+\alpha}\varphi_\alpha(\varepsilon^{-1}x)$. We define the maximal operator associated to φ_α by

$$M_{\varphi_\alpha}^+ f(x) = \sup_{\varepsilon>0} \varphi_{\alpha,\varepsilon} * |f|(x).$$

It is very easy to see that $M_{\varphi_\alpha}^+ f(x) \le C M_\alpha^+ f(x)$.

Definition 2.6. Let $b \in L^1_{loc}(\mathbb{R})$. We say that $b \in BMO$ if

$$||b||_{BMO} = \sup_I \frac{1}{|I|} \int_I |b - b_I| < \infty,$$

where I denotes any bounded interval and $b_I = \frac{1}{|I|}\int_I b$.

Definition 2.7. Let f be a locally integrable function. The one-sided sharp maximal function is defined by

$$f^{\#,+}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f \right)^+ dy.$$

It is proved in [11] that

$$\begin{aligned} f^{\#,+}(x) &\leq \sup_{h>0} \inf_{a\in\mathbb{R}} \frac{1}{h} \int_x^{x+h} (f(y)-a)^+ dy + \frac{1}{h}\int_{x+h}^{x+2h} (a - f(y))^+ dy \\ &\leq C||f||_{BMO}. \end{aligned}$$

Now we shall state our results.

Theorem 2.8. *Let $b \in BMO$, H as in (2.2) and $k = 0, 1, \ldots$. The k-th order commutator of the one-sided discrete square function is defined by*

$$S_b^{+,k} f(x) = \left|\left| \int_{\mathbb{R}} (b(x) - b(y))^k H(x-y) f(y) dy \right|\right|_{\ell^2}.$$

Then for $1 < p < \infty$ and $\omega \in A_p^+$,

$$\int_{\mathbb{R}} |S_b^{+,k} f|^p \omega \leq C \int_{\mathbb{R}} |f|^p \omega,$$

for all bounded functions f with compact support.

Theorem 2.9. *Let $0 < \alpha < 1$, $b \in BMO$, and $k = 0, 1, \ldots$. The k-th order commutator of the Weyl fractional integral is defined by*

$$I_{\alpha,b}^{+,k} f(x) = \int_x^{\infty} (b(x) - b(y))^k \frac{f(y)}{(y-x)^{1-\alpha}} dy.$$

(The Weyl fractional integral is the corresponding one for $k = 0$) Then for all $\omega \in A^+(p,q)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha$, we have

$$\left(\int_{\mathbb{R}} |I_{\alpha,b}^{+,k} f|^q \omega^q \right)^{1/q} \leq C \left(\int_{\mathbb{R}} |f|^p \omega^p \right)^{1/p},$$

for all bounded f with compact support.

In the following theorems we prove that for commutators of one-sided operators given by convolution with a smooth function, $b \in BMO$ is also a necessary condition in order to have the commutator bounded on $L^p(\omega)$.

Let $\varphi \in \mathcal{C}_c^\infty(-\infty, 0]$, $\varphi \geq 0$ nondecreasing in $(-\infty, 0]$. Then it is easy to see that there exists $C > 0$ such that $|\varphi(x-y) - \varphi(x)| \leq C|y||x|^{-2}$, for all x, y such that $|x| > 2|y|$.

For $k = 0, 1, \ldots$, the k-th order commutator of M_φ^+ with symbol b is defined by

$$M_{\varphi,b}^{+,k} f(x) = \sup_{\varepsilon>0} \int_x^{\infty} |b(x) - b(y)|^k \varphi_\varepsilon(x-y) |f(y)|\, dy.$$

Theorem 2.10. *Let $1 < p < \infty$, $b \in BMO$ and $\omega \in A_p^+$. Then*

$$\int_{\mathbb{R}} |M_{\varphi,b}^{+,k} f|^p \omega \leq C \int_{\mathbb{R}} |f|^p \omega.$$

If we consider φ as before and such that $\chi_{[-1,0]} \leq \varphi$, then we have

$$M_b^{+,k} f(x) \leq M_{\varphi,b}^{+,k} f(x),$$

where

$$M_b^{+,k} f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |b(x) - b(y)|^k |f(y)|\, dy.$$

Therefore, if $b \in BMO$, Theorem 2.10 gives that $M_b^{+,k}$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, for $1 < p < \infty$ and $\omega \in A_p^+$. In fact, more can be said, the converse is also true.

Theorem 2.11. *The following conditions are equivalent:*

(i) *$M_b^{+,k}$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, for all p with $1 < p < \infty$ and $\omega \in A_p^+$.*
(ii) *$M_b^{+,k}$ is bounded from $L^p(dx)$ to $L^p(dx)$, for some $p > 1$.*
(iii) *$b \in BMO$.*

Analogous results hold for $M_{\varphi_\alpha}^+$. Let $0 < \alpha < 1$. Suppose $\varphi_\alpha \in \mathcal{C}^\infty((-\infty, 0])$, $\varphi_\alpha \geq 0$, nondecreasing in $(-\infty, 0]$ and such that $|\varphi_\alpha(x-y) - \varphi_\alpha(x)| \leq C|y||x|^{\alpha-2}$, for all x, y such that $|x| > 2|y|$.

For $k = 0, 1, ...$, the k-th order commutator of $M_{\varphi_\alpha}^+$ with symbol b is defined by

$$M_{\varphi_\alpha,b}^{+,k} f(x) = \sup_{\varepsilon>0} \int_x^\infty |b(x) - b(y)|^k \varphi_{\alpha,\varepsilon}(x-y)|f(y)|\, dy.$$

Then we have the following:

Theorem 2.12. *Let $b \in BMO$ and let (p,q) be such that $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then, for all $\omega \in A^+(p,q)$, we have that*

$$\left(\int_{\mathbb{R}} |M_{\varphi_\alpha,b}^{+,k} f|^q \omega^q\right)^{1/q} \leq C \left(\int_{\mathbb{R}} |f|^p \omega^p\right)^{1/p}.$$

Since $M_{\varphi_\alpha}^+ f(x) \leq C M_\alpha^+ f(x)$, the case $k = 0$ is a consequence of a result of Andersen and Sawyer [2]. And if we choose φ_α such that $\chi_{[-1,0]} \leq \varphi_\alpha$, then $M_{\alpha,b}^{+,k} f(x) \leq M_{\varphi_\alpha,b}^{+,k} f(x)$, where

$$M_{\alpha,b}^{+,k} f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |b(x) - b(y)|^k |f(y)| dy.$$

For this operator we can also prove that $b \in BMO$ is a necessary condition.

Theorem 2.13. *The following conditions are equivalent:*

(i) *$M_{\alpha,b}^{+,k}$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ for all pairs (p,q) such that $\frac{1}{p} - \frac{1}{q} = \alpha$, $1 < p < q < \infty$ and $\omega \in A^+(p,q)$.*
(ii) *$M_{\alpha,b}^{+,k}$ is bounded from $L^p(dx)$ to $L^q(dx)$ for some pair (p,q) such that $\frac{1}{p} - \frac{1}{q} = \alpha$ and $1 < p < q < \infty$.*
(iii) *$b \in BMO$.*

3. Proof of the Results

The main tools for proving our results are two extrapolation theorems that appeared in [6], with slight modifications.

Theorem 3.1. *Let T be a sublinear operator defined in $\mathcal{C}_c^\infty(\mathbb{R})$. Assume that for all ω such that $\omega^{-1} \in A_1^-$, there exists $C = C(\omega)$ such that*

$$||\omega Tf||_\infty \leq C||f\omega||_\infty.$$

Then, for all $\omega \in A_p^+$, $1 < p < \infty$, there exists $C = C(\omega)$ such that

$$\left(\int |Tf|^p\omega\right)^{1/p} \leq C\left(\int |f|^p\omega\right)^{1/p},$$

provided that the left hand side is finite.

Theorem 3.2. *Let $1 < p_0 < \infty$ and T be a sublinear operator defined in $\mathcal{C}_c^\infty(\mathbb{R})$. Assume that for all $\omega \in A^+(p_0, \infty)$ there exists $C = C(\omega)$ such that*

$$||\omega Tf||_\infty \leq C||f\omega||_{p_0}.$$

Then, for all pairs (p, q) such that $1 < p < p_0$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0}$ and all $\omega \in A^+(p, q)$, there exists $C = C(\omega)$ such that

$$||\omega Tf||_q \leq C||f\omega||_p\,,$$

provided that the left hand side is finite.

We will also need the following result of Martín–Reyes and de la Torre (theorem 4 in [11]):

Theorem 3.3. *Let $1 < p < \infty$. If $\omega \in A_p^+$ and $M^+f \in L^p(\omega)$, then there exists $C = C(w)$ such that*

$$\int_{\mathbb{R}} (M^+f)^p\omega \leq C\int_{\mathbb{R}} (f^{\#,+})^p\omega.$$

An other result that will be used often is the following (see [15]):

Theorem 3.4. *Let $\omega \in A_1^-$. Then there exists $s > 1$ such that $\omega^r \in A_1^-$, for all r such that $1 < r \leq s$.*

Proof of Theorem 2.8. Let $\omega \in A_p^+$. For $b \in L^\infty \subset BMO$ and f bounded of compact support, we have that $S_b^{+,k}f \in L^p(\omega)$. Then, by Theorem 3.3,

$$\int_{\mathbb{R}} |S_b^{+,k}f|^p\omega \leq C\int_{\mathbb{R}} |M^+(S_b^{+,k}f)|^p\omega \leq C\int_{\mathbb{R}} |(S_b^{+,k}f)^{\#,+}|^p\omega. \tag{3.1}$$

To prove the theorem for any $b \in BMO$ we proceed in the same way as in [5]. We will control $(S_b^{+,k}f)^{\#,+}$ by some one-sided maximal operators. Using Theorem 3.1, we shall prove that they are bounded from $L^p(\omega)$ to $L^p(\omega)$.

Let λ be an arbitrary constant. Then $b(x)-b(y) = (b(x)-\lambda)-(b(y)-\lambda)$ and

(3.2)
$$\begin{aligned}
S_b^{+,k}f(x) &= \left\|\int_{\mathbb{R}}(b(x)-b(y))^k H(x-y)f(y)dy\right\|_{\ell^2}\\
&= \left\|\sum_{j=0}^{k} C_{j,k}(b(x)-\lambda)^j\int_{\mathbb{R}}(b(y)-\lambda)^{k-j}H(x-y)f(y)dy\right\|_{\ell^2}\\
&\le \left\|\int_{\mathbb{R}}(b(y)-\lambda)^k H(x-y)f(y)dy\right\|_{\ell^2}\\
&\quad+\left\|\sum_{j=1}^{k} C_{j,k}\ (b(x)-\lambda)^j\int_{\mathbb{R}}(b(y)-\lambda)^{k-j}H(x-y)f(y)dy\right\|_{\ell^2}\\
&= S^+((b-\lambda)^k f)(x)\\
&\quad+\left\|\sum_{j=1}^{k}\sum_{s=0}^{k-j}\ C_{j,k,s}(b(x)-\lambda)^{s+j}\int_{\mathbb{R}}(b(x)-b(y))^{k-j-s}H(x-y)f(y)dy\right\|_{\ell^2}\\
&\le S^+((b-\lambda)^k f)(x)+\sum_{m=0}^{k-1} C_{k,m}|b(x)-\lambda|^{k-m}S_b^{+,m}f(x)\,,
\end{aligned}$$

where $C_{j,k}$ (respectively $C_{j,k,s}$) are absolute constants depending only on j and k (respectively j, k and s). Let $x\in\mathbb{R}$, $h>0$. Let $i\in\mathbb{Z}$ be such that $2^i\le h<2^{i+1}$ and set $J=[x,x+2^{i+3}]$. Then, write $f=f_1+f_2$, where $f_1=f\chi_J$ and set $\lambda=b_J$. Then

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+2h}|S_b^{+,k}f(y)-S^+((b-b_J)^k f_2)(x)|dy\\
&\quad\le \frac{1}{h}\int_x^{x+2h}|S^+((b-b_J)^k f_1)(y)|dy\\
&\qquad+\frac{1}{h}\int_x^{x+2h}\left|S^+((b-b_J)^k f_2)(y)-S^+((b-b_J)^k f_2)(x)\right|dy \qquad (3.3)\\
&\qquad+\sum_{m=0}^{k-1} C_{k,m}\frac{1}{h}\int_x^{x+2h}|b(y)-b_J|^{k-m}|S_b^{+,m}f(y)|dy\\
&\quad= I(x)+II(x)+III(x).
\end{aligned}$$

Let U^+ be as in (2.1). Then

$$II(x)\le\frac{1}{h}\int_x^{x+2^{i+3}}||U^+((b-b_J)^k f_2)(y)-U^+((b-b_J)^k f_2)(x)||_{\ell^2}dy, \tag{3.4}$$

and

$$\begin{aligned}
&||U^+((b-b_J)^k f_2)(y)-U^+((b-b_J)^k f_2)(x)||_{\ell^2}\\
&\le\int_{x+2^{i+3}}^{\infty}|b(t)-b_J|^k|f(t)|||H(y-t)-H(x-t)||_{\ell^2}dt.
\end{aligned} \tag{3.5}$$

Consider the following sublinear operators defined in $\mathcal{C}_c^\infty$:

$$M_1^+ f(x) = \sup_{i\in\mathbb{Z}} \frac{1}{2^i}\int_x^{x+2^{i+2}} |S^+((b-b_J)^k f\chi_J)(y)|dy\,;$$

$$M_2^+ f(x)= \sup_{i\in\mathbb{Z}} \frac{1}{2^i}\int_x^{x+2^{i+3}}\int_{x+2^{i+3}}^\infty |b(t)-b_J|^k|f(t)|||H(y-t)-H(x-t)||_{\ell^2}dtdy\,;$$

and

$$M_{3,m}^+ f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+2h} |b(y)-b_{[x,x+8h]}|^{k-m}|f(y)|dy\,,\quad 0\le m\le k-1,$$

where, for each i, J denotes the interval $[x, x+2^{i+3}]$.

Inequalities (3.3), (3.4), (3.5) and the above definitions give that

$$(S_b^{+,k}f)^{\#,+}(x) \le C\left(M_1^+ f(x) + M_2^+ f(x) + \sum_{m=0}^{k-1} M_{3,m}^+(S_b^{+,m}f)(x)\right).$$

We shall prove, using Theorem 3.1, that these operators are bounded from $L^p(\omega)$ to $L^p(\omega)$, $\omega\in A_p^+$, $1<p<\infty$.

Boundedness of M_1^+: Let ω be a weight such that $\omega^{-1}\in A_1^-$. Let $1<q<\infty$ and $1<s<\infty$ be such that $\omega^{-qs}\in A_1^-$ (Theorem 3.4). By Hölder's and John-Nirenberg's inequalities and the fact that S^+ is bounded from $L^q(dx)$ to $L^q(dx)$ we get

(3.6)
$$\begin{aligned}
&\frac{1}{2^i}\int_x^{x+2^{i+2}} |S^+((b-b_J)^k f\chi_J)(y)|dy\\
&\le C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}} |S^+((b-b_J)^k f\chi_J)(y)|^q dy\right)^{1/q}\\
&\le C\left(\frac{1}{2^i}\int_x^{x+2^{i+3}} |(b(y)-b_J)^k f(y)|^q dy\right)^{1/q}\\
&\le C||f\omega||_\infty\left(\frac{1}{2^i}\int_x^{x+2^{i+3}} |(b(y)-b_J)^k \omega^{-1}(y)|^q dy\right)^{1/q}\\
&\le C||f\omega||_\infty\left(\frac{1}{2^i}\int_x^{x+2^{i+3}} |b-b_J|^{kqs'}\right)^{1/qs'}\left(\frac{1}{2^i}\int_x^{x+2^{i+3}} \omega^{-qs}\right)^{1/sq}\\
&\le C||f\omega||_\infty||b||_{BMO}^k\omega^{-1}(x).
\end{aligned}$$

Then, for all ω such that $\omega^{-1}\in A_1^-$,

$$||\omega M_1^+ f||_\infty \le C||b||_{BMO}^k||f\omega||_\infty\,,$$

and by Theorem 3.1, we obtain that for all $\omega\in A_p^+$, $1<p<\infty$,

$$||M_1^+ f||_{\omega,p} \le C||b||_{BMO}^k||f||_{\omega,p}\,.$$

Boundedness of M_2^+: Let ω be such that $\omega^{-1} \in A_1^-$. Then, if $I_j = [x, x+2^{j+1}]$, we have that

(3.7)
$$\begin{aligned}
&\int_{x+2^{i+3}}^{\infty} |b(t) - b_J|^k |f(t)| ||H(y-t) - H(x-t)||_{\ell^2} dt \\
&\le C \sum_{j=i+3}^{\infty} \int_{x+2^j}^{x+2^{j+1}} |b(t) - b_{I_j}|^k |f(t)| ||H(y-t) - H(x-t)||_{\ell^2} dt \\
&\quad + C \sum_{j=i+3}^{\infty} |b_{I_j} - b_J|^k \int_{x+2^j}^{x+2^{j+1}} |f(t)| ||H(y-t) - H(x-t)||_{\ell^2} dt \\
&= IV(x) + V(x).
\end{aligned}$$

We choose (s, s') and (t, t') such that $\omega^{-s} \in A_1^-$ and $\omega^{-st'} \in A_1^-$. Then, by Hölder's inequality with exponents (s, s') and (t, t'),

$$\begin{aligned}
IV(x) &\le C \sum_{j=i+3}^{\infty} \left(\int_{I_j} |b - b_{I_j}|^{ks} \omega^{-s} \omega^{s} |f|^s \right)^{1/s} \\
&\qquad \times \left(\int_{x+2^j}^{x+2^{j+1}} ||H(y-t) - H(x-t)||_{\ell^2}^{s'} dt \right)^{1/s'} \\
&\le C ||f\omega||_\infty \sum_{j=i+3}^{\infty} \left(\int_{I_j} |b - b_{I_j}|^{kst} \right)^{1/st} \left(\int_{I_j} \omega^{-st'} \right)^{1/st'} \\
&\qquad \times \left(\int_{x+2^j}^{x+2^{j+1}} ||H(y-t) - H(x-t)||_{\ell^2}^{s'} dt \right)^{1/s'} .
\end{aligned} \tag{3.8}$$

It is proved in theorem 1.6 of [19] that for all $y \in [x, x+2^{i+3}]$ the kernel H satisfies

$$\left(\int_{x+2^j}^{x+2^{j+1}} ||H(y-t) - H(x-t)||_{\ell^2}^{s'} dt \right)^{1/s'} \le C \frac{2^{i/s'}}{2^j}. \tag{3.9}$$

Then, using that $b \in BMO$, the fact that $\omega^{-st'} \in A_1^-$ and (3.9), we get

$$\begin{aligned}
IV(x) &\le C ||f\omega||_\infty \sum_{j=i+3}^{\infty} ||b||_{BMO}^k (2^j)^{1/st} \omega^{-1}(x) (2^{j+1})^{1/st'} \frac{2^{i/s'}}{2^j} \\
&\le C\omega^{-1}(x) ||f\omega||_\infty ||b||_{BMO}^k \sum_{j=i+3}^{\infty} \left(\frac{2^i}{2^j} \right)^{1/s'} \\
&\le C\omega^{-1}(x) ||f\omega||_\infty ||b||_{BMO}^k .
\end{aligned} \tag{3.10}$$

Using again Hölder's inequality, (3.9), the fact that $w^{-s} \in A_1^-$ and lemma

1 in [5], we get

$$
\begin{aligned}
V(x) &= C\sum_{j=i+3}^{\infty}|b_{I_j}-b_J|^k\int_{x+2^j}^{x+2^{j+1}}|f(t)|||H(y-t)-H(x-t)||_{\ell^2}dt\\
&\le C\sum_{j=i+3}^{\infty}||b||^k_{BMO}(2(j-i))^k||f\omega||_\infty\left(\int_{I_j}\omega^{-s}\right)^{1/s}\\
&\qquad\times\left(\int_{x+2^j}^{x+2^{j+1}}||H(y-t)-H(x-t)||^{s'}_{\ell^2}dt\right)^{1/s'} \qquad (3.11)\\
&\le C||b||^k_{BMO}||f\omega||_\infty\omega^{-1}(x)\sum_{j=i+3}^{\infty}(2(j-i))^k(2^{j+1})^{1/s}\frac{2^{i/s'}}{2^j}\\
&\le C\omega^{-1}(x)||b||^k_{BMO}||f\omega||_\infty\sum_{j=i+3}^{\infty}\frac{(j-i)^k2^{i/s'}}{2^{j/s'}}\\
&\le C\omega^{-1}(x)||b||^k_{BMO}||f\omega||_\infty.
\end{aligned}
$$

Then, by (3.7), (3.10) and (3.11), we get that, for all ω such that $\omega^{-1}\in A_1^-$,

$$||\omega M_2^+f||_\infty\le C||f\omega||_\infty.$$

Then, by Theorem 3.1, for all $1<p<\infty$ and $\omega\in A_p^+$,

$$||M_2^+f||_{p,\omega}\le C||f||_{p,\omega}\,.$$

Boundedness of $M^+_{3,m}$: Let ω be such that $\omega^{-1}\in A_1^-$ and let $q>1$ be such that $\omega^{-q}\in A_1^-$. Then, using again Hölder's and John-Nirenberg's inequalities we obtain

$$
\begin{aligned}
&\frac{1}{h}\int_x^{x+2h}|b(y)-b_{[x,x+8h]}|^{k-m}|f(y)|dy\\
&\le\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_{[x,x+8h]}|^{(k-m)q'}dy\right)^{1/q'}\left(\frac{1}{h}\int_x^{x+2h}|f(y)|^qdy\right)^{1/q}\\
&\le C||b||^{k-m}_{BMO}\left(\frac{1}{h}\int_x^{x+2h}|f(y)|^qdy\right)^{1/q}\\
&\le C||b||^{k-m}_{BMO}||f\omega||_\infty\left(\frac{1}{h}\int_x^{x+2h}\omega^{-q}dy\right)^{1/q}\\
&\le C||b||^{k-m}_{BMO}||f\omega||_\infty\omega^{-1}(x).
\end{aligned}
$$

Then, for all ω such that $\omega^{-1}\in A_1^-$,

$$||\omega M^+_{3,m}f||_\infty\le C||b||^{k-m}_{BMO}||\omega f||_\infty.$$

Therefore, by Theorem 3.1, we have that, for all $\omega \in A_p^+$, $1 < p < \infty$,

$$||M_{3,m}^+ f||_{\omega,p} \le C||b||_{BMO}^{k-m}||f||_{\omega,p}.$$

Using now the induction principle (the case $k = 0$ was proved in [19]), we obtain that, for all $\omega \in A_p^+$, $1 < p < \infty$,

$$||M_{3,m}^+(S_b^{+,m} f)||_{\omega,p} \le C||b||_{BMO}^{k-m}||S_b^{+,m} f||_{\omega,p} \le C||b||_{BMO}^{k}||f||_{\omega,p}. \qquad \square$$

Proof of Theorem 2.9. This proof follows the same pattern as the preceding one. Let $b \in BMO$ bounded, and $\lambda \in \mathbb{R}$, then, as in (3.2), we can write

$$I_{\alpha,b}^{k,+} f(x) = I_\alpha^+((b-\lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m}(b(x)-\lambda)^{k-m} I_{\alpha,b}^{+,m} f(x).$$

Let $x \in \mathbb{R}$, $h > 0$ and $J = [x, x+4h]$. Write $f = f_1 + f_2$, where $f_1 = f\chi_J$ and set $\lambda = b_J$. Then,

(3.12)

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+2h} \left| I_{\alpha,b}^{+,k} f(y) - I_\alpha^+((b-b_J)^k f_2)(x+2h)\right| dy \\
&\le \frac{1}{h}\int_x^{x+2h} |I_\alpha^+((b-b_J)^k f_1)(y)| dy \\
&\quad + \frac{1}{h}\int_x^{x+2h} \left| I_\alpha^+((b-b_J)^k f_2)(y) - I_\alpha^+((b-b_J)^k f_2)(x+2h)\right| dy \\
&\quad + \sum_{m=0}^{k-1} C_{k,m}\frac{1}{h}\int_x^{x+2h} |b(y)-b_J|^{k-m}|I_{\alpha,b}^{+,m} f(y)| dy \\
&= I(x) + II(x) + III(x).
\end{aligned}$$

It is clear that

$$III(x) \le \sum_{m=0}^{k-1} C_{k,m} M_{3,m}^+(I_{\alpha,b}^{+,m} f)(x),$$

where $M_{3,m}^+$ is as in the proof of Theorem 2.8. Then, we already know that $M_{3,m}^+$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, provided $\omega \in A_p^+$ and $1 < p < \infty$. So, if $\omega \in A^+(p,q)$, $1/p - 1/q = \alpha$, then $\omega^q \in A_q^+$, and by induction (see k=0 in [10]), we obtain

$$||M_{3,m}^+(I_{\alpha,b}^{+,m} f)||_{\omega^q,q} \le C||b||_{BMO}^{k-m}||I_{\alpha,b}^{+,m} f||_{\omega^q,q} \le C||b||_{BMO}^{k}||f||_{\omega^p,p},$$

for all $f \in \mathcal{C}_c^\infty(\mathbb{R})$.

To control $I(x)$ let us define

$$M_4^+ f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+2h} |I_\alpha^+((b-b_J)^k f\chi_J)(y)| dy,$$

where, for each $h>0$, $J=[x,x+4h]$. It is not difficult to see that M_4^+ is a sublinear operator in $\mathcal{C}_c^\infty$ and it is clear that $I(x)\le M_4^+f(x)$. Let us prove that M_4^+ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ using Theorem 3.2.

Let $\omega\in A^+(\frac{1}{\alpha},\infty)$, then $\omega^{\frac{-1}{1-\alpha}}\in A_1^-$. Therefore, there exist $t_1>1$ and $t_2>1$ such that $\omega^{\frac{-t_1}{1-\alpha}}\in A_1^-$ and $\omega^{\frac{-t_1t_2}{1-\alpha}}\in A_1^-$. Let $s>1$ and $r>1$ be such that $s=t_1/(1-\alpha)$ and $1/r-1/s=\alpha$. Then, using Hölder's inequality and the fact that I_α^+ is bounded from $L^r(\mathbb{R})$ to $L^s(\mathbb{R})$, we get

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+2h}|I_\alpha^+((b-b_J)^kf\chi_J)(y)|dy\\
&\le\left(\frac{1}{h}\int_x^{x+2h}|I_\alpha^+((b-b_J)^kf\chi_J)(y)|^sdy\right)^{1/s}\\
&\le Ch^\alpha\left(\frac{1}{h}\int_x^{x+4h}|(b(y)-b_J)^kf(y)|^r\omega^r\omega^{-r}dy\right)^{1/r}\\
&\le Ch^\alpha\left(\frac{1}{h}\int_x^{x+4h}|(b-b_J)^k|^{r\frac{s}{r}}\omega^{-r\frac{s}{r}}\right)^{\frac{1}{s}}\left(\frac{1}{h}\int_x^{x+4h}|f|^{\frac{1}{\alpha}}\omega^{\frac{1}{\alpha}}\right)^{\alpha}.
\end{aligned}\tag{3.13}$$

Therefore, using Hölder's and John-Nirenberg's inequalities and the fact that $\omega^{-st_2}\in A_1^-$, the chain of inequalities in (3.13) can be continued as follows:

$$\begin{aligned}
&\le C||f\omega||_{\frac{1}{\alpha}}\left(\frac{1}{h}\int_x^{x+4h}|b-b_J|^{ks}\omega^{-s}\right)^{\frac{1}{s}}\\
&\le C||f\omega||_{\frac{1}{\alpha}}\left(\frac{1}{h}\int_x^{x+4h}|b-b_J|^{kst_2'}\right)^{\frac{1}{st_2'}}\left(\frac{1}{h}\int_x^{x+4h}\omega^{-st_2}\right)^{\frac{1}{st_2}}\\
&\le C||f\omega||_{\frac{1}{\alpha}}||b||_{BMO}^k\omega^{-1}(x).
\end{aligned}\tag{3.14}$$

As a consequence,

$$||\omega M_4^+f||_\infty\le C||b||_{BMO}^k||f\omega||_{\frac{1}{\alpha}}.$$

Then, by Theorem 3.2, for all $\omega\in A^+(p,q)$, $\frac{1}{p}-\frac{1}{q}=\alpha$,

$$||M_4^+f||_{\omega^q,q}\le C||b||_{BMO}^k||f||_{\omega^p,p}.$$

Finally, we shall estimate $II(x)$. We have that

$$II(x)=\frac{1}{h}\int_x^{x+2h}\left|\int_{x+4h}^\infty\sigma(t,y)dt\right|dy,$$

where

$$\sigma(t,y)=(b(t)-b_J)^kf(t)\left(\frac{1}{(y-t)^{1-\alpha}}-\frac{1}{(x+2h-t)^{1-\alpha}}\right).$$

Consider the following sublinear operator in $C_c^\infty(\mathbb{R})$:

$$M_5^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+2h} \left| \int_{x+4h}^{\infty} \sigma(t,y)dt \right| dy.$$

For each $j \in \mathbb{N}$, set $I_j = [x, x+2^{j+1}h]$. Then,

$$\begin{aligned}
&\frac{1}{h} \int_x^{x+2h} \left| \int_{x+4h}^{\infty} \sigma(t,y)dt \right| dy \\
&\le \frac{1}{h} \int_x^{x+2h} \sum_{j=2}^{\infty} \int_{x+2^j h}^{x+2^{j+1}h} |\sigma(t,y)|\, dtdy \\
&\le C \sum_{j=2}^{\infty} \int_{x+2^j h}^{x+2^{j+1}h} |b(t)-b_J|^k |f(t)| \frac{2h}{(h(2^j-2))^{2-\alpha}} dt \\
&\le C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} \left(\frac{2}{2^{j-1}h} \int_{x+2^j h}^{x+2^{j+1}h} |b(t)-b_J|^k |f(t)| dt \right) \qquad (3.15) \\
&\le C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} \left(\frac{2}{2^{j-1}h} \int_{x+2^j h}^{x+2^{j+1}h} |b(t)-b_{I_j}|^k |f(t)| dt \right) \\
&\qquad + C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} \left(\int_{x+2^j h}^{x+2^{j+1}h} |b_{I_j}-b_J|^k |f(t)| dt \right) \\
&\le C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} (IV(x)+V(x)).
\end{aligned}$$

Let $\omega \in A^+(\frac{1}{\alpha},\infty)$. Then $\omega^{\frac{-1}{1-\alpha}} \in A_1^-$. Choose $r>1$ such that $\omega^{\frac{-r}{1-\alpha}} \in A_1^-$. Then, by Hölder's and John-Nirenberg's inequalities,

$$\begin{aligned}
IV(x) &\le \left(\frac{2}{2^{j-1}h} \int_{x+2^j h}^{x+2^{j+1}h} |f|^{\frac{1}{\alpha}} \omega^{\frac{1}{\alpha}} \right)^{\alpha} \\
&\quad \times \left(\frac{2}{2^{j-1}h} \int_{x+2^j h}^{x+2^{j+1}h} |b(t)-b_{I_j}|^{\frac{k}{1-\alpha}} \omega^{\frac{-1}{1-\alpha}} \right)^{1-\alpha} \\
&\le C(2^j h)^{-\alpha} ||f\omega||_{\frac{1}{\alpha}} \left(\frac{1}{2^j h} \int_x^{x+2^{j+1}h} |b(t)-b_{I_j}|^{\frac{kr'}{1-\alpha}} \right)^{\frac{1-\alpha}{r'}} \qquad (3.16) \\
&\quad \times \left(\frac{1}{2^j h} \int_x^{x+2^{j+1}h} \omega^{\frac{-r}{1-\alpha}} \right)^{\frac{1-\alpha}{r}} \\
&\le C||b||_{BMO}^k (2^j h)^{-\alpha} ||f\omega||_{\frac{1}{\alpha}} \omega^{-1}(x).
\end{aligned}$$

Using again lemma 1 in [5] and Hölder's inequality,

$$\begin{aligned} V(x) &\leq \frac{1}{2^j h}|b_{I_j}-b_J|^k \int_x^{x+2^{j+1}h} |f(t)|dt \\ &\leq C(2j)^k ||b||_{BMO}^k \left(\frac{1}{2^j h}\int_x^{x+2^{j+1}h} |f|^{\frac{1}{\alpha}}\omega^{\frac{1}{\alpha}}\right)^{\alpha} \\ &\qquad \times \left(\frac{1}{2^j h}\int_x^{x+2^{j+1}h} \omega^{\frac{-1}{1-\alpha}}\right)^{1-\alpha} \\ &\leq C(2j)^k (2^j h)^{-\alpha} ||b||_{BMO}^k ||f\omega||_{\frac{1}{\alpha}} \omega^{-1}(x). \end{aligned} \tag{3.17}$$

Putting together inequalities (3.15), (3.16) and (3.17), we get that

$$\begin{aligned} &\frac{1}{h}\int_x^{x+2h} \left|\int_{x+4h}^{\infty} \sigma(t,y)dt\right| dy \\ &\leq C||b||_{BMO}^k ||f\omega||_{\frac{1}{\alpha}} \omega^{-1}(x) \sum_{j=2}^{\infty} \left(\frac{2^{1-\alpha}}{2^j} + \frac{2^{1-\alpha}(2j)^k}{2^j}\right) \\ &\leq C||b||_{BMO}^k ||f\omega||_{\frac{1}{\alpha}} \omega^{-1}(x). \end{aligned}$$

Taking supremums first on $h > 0$ and then on $x \in \mathbb{R}$, we get

$$||\omega M_5^+ f||_\infty \leq C||b||_{BMO}^k ||f\omega||_{\frac{1}{\alpha}} \,.$$

So, by Theorem 3.2, for all $\omega \in A^+(p,q)$, $\frac{1}{p} - \frac{1}{q} = \alpha$,

$$||M_5^+ f||_{\omega^q,q} \leq C||b||_{BMO}^k ||f||_{\omega^p,p} \,.$$

□

Proof of Theorem 2.10.

This proof also follows the same pattern as the preceding ones. Let $b \in BMO$ bounded and let $\lambda \in \mathbb{R}$. Then, as in (3.2), we have

$$M_{\varphi,b}^{+,k} f(x) \leq M_\varphi^+((b-\lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m}|b(x)-\lambda|^{k-m} M_{\varphi,b}^{+,m} f(x).$$

Let us fix $x \in \mathbb{R}$ and $h > 0$ and let $J = [x, x+8h]$. Write $f = f_1 + f_2$, where $f_1 = f\chi_J$, and also write $\lambda = b_J$. Then, as in (3.3) and (3.12), it follows that

$$\begin{aligned} &\frac{1}{h}\int_x^{x+2h} \left|M_{\varphi,b}^{+,k} f(y) - M_\varphi^+((b-b_J)^k f_2)(x+2h)\right| dy \\ &\leq \frac{1}{h}\int_x^{x+2h} |M_\varphi^+((b-b_J)^k f_1)(y)|dy \\ &\qquad + \frac{1}{h}\int_x^{x+2h} \left|M_\varphi^+((b-b_J)^k f_2)(y) - M_\varphi^+((b-b_J)^k f_2)(x+2h)\right| dy \\ &\qquad + \sum_{m=0}^{k-1} C_{k,m}\frac{1}{h}\int_x^{x+2h} |b(y)-b_J|^{k-m} |M_{\varphi,b}^{+,m} f(y)|dy \\ &= I(x) + II(x) + III(x). \end{aligned}$$

It is clear that $III(x) \leq \sum_{m=0}^{k-1} C_{k,m} M^+_{3,m}(M^{+,m}_{\varphi,b} f)(x)$, being $M^+_{3,m}$ the same as in the proofs of Theorems 2.8 and 2.9, and

$$II(x) \leq C\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^{\infty} \frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J|^k|f(z)|dzdy, \tag{3.18}$$

by the conditions imposed on the kernel φ.

Consider the following sublinear operators defined on $\mathcal{C}^\infty_c$:

$$M^+_6 f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+2h} |M^+_\varphi((b-b_J)^k f\chi_J)(y)|dy$$

and

$$M^+_7 f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^{\infty} \frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J|^k|f(z)|dzdy,$$

where, for each $h > 0$, J is the interval $[x, x+8h]$.

The above inequalities and definitions give that

$$(M^{+,k}_{\varphi,b} f)^{\#,+}(x) \leq C\left(M^+_6 f(x) + M^+_7 f(x) + \sum_{m=0}^{k-1} M^+_{3,m}(M^{+,m}_{\varphi,b} f)(x)\right). \tag{3.19}$$

Let $\omega \in A^+_p$, $1 < p < \infty$. Then, acting as in the boundedness of M^+_1, in the proof of Theorem 2.8, we get that M^+_6 is bounded from $L^p(\omega)$ to $L^p(\omega)$. On the other hand, we already know that $M^+_{3,m}$ is bounded from $L^p(\omega)$ to $L^p(\omega)$ and, to proceed with the induction, we observe that the case $k = 0$ is a consequence of the fact that $M^+_\varphi f$ is pointwise equivalent to $M^+ f$. Therefore, we only have to prove that M^+_7 is bounded from $L^p(\omega)$ to $L^p(\omega)$, for $\omega \in A^+_p$, $1 < p < \infty$.

Let us use again Theorem 3.1. Let ω be such that $\omega^{-1} \in A^-_1$. For each $j \in \mathbb{N}$, let $I_j = [x, x+2^j h]$. Then

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^{\infty} \frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J|^k|f(z)|dzdy \\
&\leq C\frac{1}{h}\int_x^{x+2h} h\sum_{j=3}^{\infty}\int_{x+2^j h}^{x+2^{j+1}h} \frac{|b(z)-b_J|^k}{(z-(x+2h))^2}|f(z)|dzdy \\
&\leq Ch||f\omega||_\infty \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j-2)^2h^2}\frac{1}{2^{j+1}}\int_{I_j} |b(z)-b_J|^k\omega^{-1}(z)dz \\
&\leq C||f\omega||_\infty \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j-2)^2}\left(\frac{1}{2^{j+1}h}\int_{I_j} |b(z)-b_{I_j}|^k\omega^{-1}(z)dz\right. \\
&\qquad\left. + \frac{1}{2^{j+1}h}\int_{I_j} |b_{I_j}-b_J|^k\omega^{-1}(z)dz\right) \\
&= C||f\omega||_\infty \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j-2)^2}\left(IV(x)+V(x)\right).
\end{aligned} \tag{3.20}$$

Let $q>1$ be such that $\omega^{-q}\in A_1^-$ (Theorem 3.4). Then, Hölder's and John-Nirenberg's inequalities give
(3.21)

$$\begin{aligned}IV(x)&\le\left(\frac{1}{2^{j+1}h}\int_x^{x+2^{j+1}h}|b-b_{I_j}|^{kq'}\right)^{1/q'}\left(\frac{1}{2^{j+1}h}\int_x^{x+2^{j+1}h}\omega^{-q}\right)^{1/q}\\&\le C||b||_{BMO}^k\omega^{-1}(x).\end{aligned}$$

In [5], it is proved that $|b_{I_j}-b_J|^k\le C|2j|^k||b||_{BMO}^k$ for all $3\le j$. Then, using this fact and using that $\omega^{-1}\in A_1^-$, we obtain

$$V(x)=\frac{1}{2^{j+1}h}\int_{I_j}|b_{I_j}-b_J|^k\omega^{-1}\le C|2j|^k||b||_{BMO}^k\omega^{-1}(x). \tag{3.22}$$

Then by (3.20), (3.21) and (3.22) we get that, for all ω such that $\omega^{-1}\in A_1^-$,

$$||\omega M_7^+f||_\infty\le C||b||_{BMO}^k||f\omega||_\infty,$$

and, by Theorem 3.1, for all $\omega\in A_p^+$, $1<p<\infty$,

$$||M_7^+f||_{\omega,p}\le C||b||_{BMO}^k||f||_{\omega,p}\,.$$

Now, we only have to take into account (3.19) and Theorem 3.3 to conclude the proof of Theorem 2.10. □

Remark 3.5. Following the same steps as in the proof of Theorem 2.10 we get Corollary 1 of [5], i.e., the boundedness of the commutator of one-sided singular integrals (introduced in [1]), from $L^p(\omega)$ to $L^p(\omega)$, $\omega\in A_p^+$, $1<p<\infty$.

Proof of Theorem 2.11.
$(iii)\Rightarrow(i)$ It is a consequence of Theorem 2.10.
$(i)\Rightarrow(ii)$ Let $p>1$ and set $\omega\equiv 1$.
$(ii)\Rightarrow(iii)$ Set $I=(a,b)$, $I^+=(b,c)$, and $|I|=|I^+|$. Then

$$\begin{aligned}\frac{1}{|I|}\int_I|b(y)-b_{I^+}|dy&\le\left(\frac{1}{|I|}\int_I|b(y)-b_{I^+}|^kdy\right)^{1/k}\\&=\left(\frac{1}{|I|}\int_I\left|\frac{1}{|I^+|}\int_{I^+}(b(y)-b(x))dx\right|^kdy\right)^{1/k}\\&\le\left(\frac{1}{|I|}\int_I\left(\frac{1}{|I^+|}\int_{I^+}|b(y)-b(x)|^kdx\right)dy\right)^{1/k}.\end{aligned}$$

Observe that, for $y\in I$,

$$\begin{aligned}\frac{1}{|I^+|}\int_{I^+}|b(x)-b(y)|^kdx&=\frac{1}{|I^+|}\int_y^c|b(x)-b(y)|^k\chi_{I^+}(x)dx\\&\le CM_b^{+,k}\chi_{I^+}(y).\end{aligned}$$

Then, by Hölder's inequality and (ii),

$$\begin{aligned}\frac{1}{|I|}\int_I |b(y)-b_{I^+}|dy &\le C\left(\frac{1}{|I|}\int_I M_b^{+,k}\chi_{I^+}(y)dy\right)^{1/k}\\ &\le C\left(\frac{1}{|I|}\int_I |M_b^{+,k}\chi_{I^+}(y)|^p dy\right)^{1/pk}\\ &\le C\left(\frac{1}{|I|}\int_{\mathbb{R}} |\chi_{I^+}(y)|^p dy\right)^{1/pk} \le C\left(\frac{|I^+|}{|I|}\right)^{1/pk} = C.\end{aligned}$$

So $b\in BMO$. $\square$

Proof of Theorem 2.12. Observe that by the conditions given on φ_α, $M^{+,k}_{\varphi_\alpha,b}$ can be treated in the same way as $I^{+,k}_{\alpha,b}$, the commutator of the one-sided fractional operator. Observe that the case $k=0$ is a consequence of the fact that $M^+_{\varphi_\alpha}f(x)\le CM^+_\alpha f(x)$ and the result in [2]. $\square$

Proof of Theorem 2.13.
$(iii)\Rightarrow(i)$ It is a consequence of Theorem 2.12.
$(i)\Rightarrow(ii)$ Given an appropriate pair (p,q), set $\omega\equiv 1$.
$(ii)\Rightarrow(iii)$ Set $I=(a,b)$, $I^+=(b,c)$, and $|I|=|I^+|$. Then

$$\begin{aligned}\frac{1}{|I|}\int_I |b-b_{I^+}|dy &\le \left(\frac{1}{|I|}\int_I\left(\frac{1}{|I^+|}\int_{I^+}|b(x)-b(y)|^k dx\right)dy\right)^{1/k}\\ &= \left(\frac{|I|^{-\alpha}}{|I|}\int_I\left(\frac{1}{|I^+|^{1-\alpha}}\int_{I^+}|b(x)-b(y)|^k dx\right)dy\right)^{1/k}.\end{aligned}$$

Observe that, for $y\in I$,

$$\begin{aligned}\frac{1}{|I^+|^{1-\alpha}}\int_{I^+}|b(x)-b(y)|^k dx &= \frac{1}{|I^+|^{1-\alpha}}\int_y^c |b(x)-b(y)|^k\chi_{I^+}(x)dx\\ &\le CM^{+,k}_{\alpha,b}\chi_{I^+}(y).\end{aligned}$$

Then, by Hölder's inequality and (ii),

$$\begin{aligned}\frac{1}{|I|}\int_I |b-b_{I^+}|dy &\le C\left(\frac{|I|^{-\alpha}}{|I|}\int_I M^{+,k}_{\alpha,b}\chi_{I^+}(y)dy\right)^{\frac{1}{k}}\\ &\le C\left(|I|^{-\alpha}\left(\frac{1}{|I|}\int_I |M^{+,k}_{\alpha,b}\chi_{I^+}(y)|^q dy\right)^{\frac{1}{q}}\right)^{\frac{1}{k}}\\ &\le C\left(|I|^{-\alpha-\frac{1}{q}}\left(\int_{\mathbb{R}}|\chi_{I^+}(y)|^p dy\right)^{\frac{1}{p}}\right)^{\frac{1}{k}}\\ &\le C\left(|I|^{-\alpha-\frac{1}{q}+\frac{1}{p}}\right)^{1/k} = C.\end{aligned}$$

So $b\in BMO$. $\square$

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References

[1] H. Aimar, L. Forzani and F.J. Martín-Reyes, *On weighted inequalities for one-sided singular integrals*, Proc. Amer. Math. Soc. **125** (1997), 2057-2064.

[2] K. F. Andersen and E. T. Sawyer, *Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators*, Trans. Amer. Math. Soc. **308** (1988), no. 2, 547–558.

[3] S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. **292** (1985), 103-122.

[4] R. Coifman, R. Rochberg, G.Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103 (2)** (1976), 611-635.

[5] M. Lorente and M.S. Riveros, *Weighted inequalities for commutators of one-sided singular integrals*, Comment. Math. Univ. Carolinae **43 (1)** (2002), 83-101.

[6] R. Macías and M.S. Riveros, *One-sided extrapolation at infinity and singular integrals*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 1081-1102.

[7] F.J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), 691-698.

[8] F.J. Martín-Reyes, P. Ortega and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. **319 (2)** (1990), 517-534.

[9] F.J. Martín-Reyes, L. Pick and A. de la Torre, *A_∞^+ condition,*, Canad. J. Math. **45** (1993), 1231-1244.

[10] F.J. Martín-Reyes and A. de la Torre, *Two weighted inequalities for fractional one-sided maximal operators*, Proc. Amer. Math. Soc. **117 (2)** (1993), 483-489.

[11] ______, *One-sided BMO spaces*, J. London Math. Soc. 2 **49** (1994), 529-542.

[12] B. Muckenhoupt, *Weighted norm inequalities for classical operators*, Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, R.I. **35** (1979), no. 1, 69–83.

[13] C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128 (1)** (1995), 163-185.

[14] ______, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. Appl. **3 (6)** (1997), 743-756.

[15] E. T. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53-61.

[16] C. Segovia and J.L. Torrea, *Vector-valued commutators and applications*, Indiana Univ. Math. J. **38 (4)** (1989), 959-971.

[17] ______, *Weighted inequalities for commutators of fracctional and singular integrals.*, Publ. Mat. **35** (1991), 209-235.

[18] ______, *Higher order commutators for vector-valued Calderón-Zygmund operators.*, Trans. Amer. Math. Soc. **336 (2)** (1993), 537-556.

[19] A. de la Torre and J.L. Torrea, *One-sided discrete square function*, Studia Math. **156 (3)** (2003), 243-260.

Análisis Matemático. Facultad de Ciencias. Universidad de Málaga. (29071) Málaga, Spain

E-mail address: lorente@anamat.cie.uma.es

FaMAF, Universidad Nacional de Córdoba, CIEM–Conicet. (5000) Córdoba, Argentina.

E-mail address: sriveros@mate.uncor.edu