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ON HÖRMANDER'S CONDITION FOR SINGULAR INTEGRALS

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1. Introduction

In this note we present some results showing how singular integrals are controlled by maximal operators. The proofs will appear elsewhere ([4])

We start with some basic definitions:

Definition 1.1.
- *Mf will be the Hardy-Littlewood Maximal Operator.*
- *For $1 < t$, $M_t f(x) = (M|f|^t(x))^{\frac{1}{t}}$.*
- *A singular integral will be an operator T of the form*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y) f(y)\, dy,$$

  which is bounded on some L^p, $\quad 1 < p < \infty$.
- *A weight w is locally integrable nonnegative function.*
- *The class A_p, $p > 1$, will denote the class of all weights w for which the following inequality holds:*

$$\int (Mf)^p w \leq C \int |f|^p w$$

  while A_1 is the set of weights w such that

$$\int_{\{x: Mf(x) > d\}} w \leq \frac{C}{d} \int |f| w.$$

- *$A_\infty = \cup_{p \geq 1} A_p$.*

Remark 1.2.
- *Clearly $Mf(x) \leq M_t f(x) \leq M_r f(x) \quad t < r$ for any f and x.*
- *An important property of A_p $p > 1$, is that $w \in A_p$ implies that there exists $1 < q < p$ so that $w \in A_q$.*

In dimension one there is a theory of weights for the one-sided Hardy-Littlewood Maximal Operator

$$M^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_x^{x+h} |f|$$

started by E. Sawyer.(See [11], [6] and [7].)

Definition 1.3.
- *A_p^+ is the class of weights for which M^+ maps $L^p(w)$ boundedly to itself.*

- A_1^+ *is the class of weights for which*

$$\int_{\{x:M^+f(x)>d\}} w \leq \frac{C}{d}\int |f|w.$$

- A_∞^+ *is the union of the* A_p^+ *for* $1 < p$

Remark 1.4. *Any increasing function is in* A_1^+. *Since it is easy to see that this is not true for the weights in* A_p, *it follows* $A_p^+ \supsetneq A_p$

A classical result of Coifman (see [3],) states that if the kernel K satisfies the following condition:

There exists $\alpha > 0$, $C > 0$ and $c > 1$ such that

$$|K(x-y) - K(-y)| \leq C\frac{|x|^\alpha}{|y|^{\alpha+n}}, \quad \text{if } \; |y| > c|x| \quad (L)$$

then, for any $0 < p < \infty$ and $w \in A_\infty$, there exists a constant C such that

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (Mf(x))^p w(x)\, dx, \quad (C)$$

for any f for which the left hand side is finite.

From this result one can obtain the boundedness of the operator T in $L^p(w)$ for $w \in A_p$. This means that (L) is enough to obtain the weighted theory for T. But it is well known that in order to prove the boundedness of T in $L^p(dx)$ it is enough for the kernel to satisfy a weaker condition, namely the Hörmander condition.

$$\sup_{x\in\mathbb{R}^n} \int_{|y|>c|x|} |K(x-y) - K(-y)|\, dy < \infty. \quad (H)$$

For a long time it has been an open problem whether it was posible to obtain Coifman's result assuming only Hörmander's condition.

Recently Martell, Pérez and Trujillo ([5]) have proved that it is not possible. Actually they prove that (C) fails even if K satisfies certain conditions which are weaker than (L) but stronger than (H).

Definition 1.5. *Let* $1 \leq r \leq \infty$, *we say that the kernel* K *satisfies the* L^r-*Hörmander condition if there exists* $c_r > 1$ *and* $C_r > 0$, *such that for any* $x \in \mathbb{R}^n$ *and* $R > c_r|x|$

$$\sum_{m=1}^{\infty} (2^m R)^n \quad \left(\tfrac{1}{(2^m R)^n}\int_{2^m R<|y|\leq 2^{m+1}R} |K(x-y) - K(-y)|^r\, dy\right)^{\frac{1}{r}} \leq C_r,$$

if $\quad r < \infty$

and

$$\sum_{m=1}^{\infty} (2^m R)^n \sup_{2^m R<|y|\leq 2^{m+1}R} |K(x-y) - K(-y)| \leq C_\infty, \quad \textit{if} \quad r = \infty.$$

We will denote by H_r the class of kernels satisfying this condition. It is clear that the classes are nested.

$$L \equiv H^* \subset H_\infty \subset H_r \subset H_s \subset H_1 \equiv H, \quad 1 < s < r.$$

It has been known for some time that if $K \in H_r$, then T can be controlled by a maximal operator.

Theorem (See [9]). *Let $1 < r \leq \infty$ and T a singular integral whose kernel K belongs to H_r, then for any $0 < p < \infty$ and $w \in A_\infty$ there exists a constant C such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_{r'} f(x))^p\, w(x)\, dx, \tag{1.6}$$

whenever the left hand side is finite

The result of Martell, Pérez and Trujillo is that it is not possible to improve (1.6).

Theorem.(See [5]) *Let $1 \leq r < \infty$ y $1 \leq t < r'$. there exists a singular integral T, whose kernel K belongs to H_r, such that for any $0 < p < \infty$,* **it is not true** *that for any $w \in A_\infty$*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_t f(x))^p\, w(x)\, dx,$$

whenever the left hand side is finite.

This theorem leaves open the following question:

¿ What happens between H_∞ and the intersection of the H_r, $1 \leq r < \infty$?.
More precisely:

1.) Is $\cap H_r \setminus H_\infty$ nonempty?

If this is the case:

2.) What can be said about the singular integrals with kernels in $\cap H_r \setminus H_\infty$?

Obviously for those kernels one has

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_t f(x))^p\, w(x)\, dx, \quad \text{for any} \quad 1 < t. \tag{1.7}$$

But since $K \notin H_\infty$ we may not assert that

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (Mf(x))^p\, w(x)\, dx. \tag{1.8}$$

This does not mean that we might not be able to improve (1.8) obtaining some estimates of the type,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} (M_{\mathcal{A}} f(x))^p\, w(x)\, dx, \tag{1.9}$$

where $M_{\mathcal{A}}$ is a maximal operator such that $Mf(x) \leq M_{\mathcal{A}} f(x) \leq M_t f(x)$ for any f, x y $1 < t$.

2. Main Results

Let us start with the first question

Theorem 2.10. *There exists a vector valued operator U bounded in every L^p, $1 < p < \infty$, whose kernel K is in $\cap H_r \setminus H_\infty$*

There is no need to construct such example. It is in the literature (See [10]). Le us recall its definition:

Definition 2.11. *Let f be any measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ we consider the average*

$$A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f = \int_{\mathbb{R}} \frac{1}{2^n} \chi_{(-2^n,0)}(x-y) f(y) dy.$$

We define the square function Sf by:

$$Sf(x) = \left(\sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{\frac{1}{2}}.$$

This is not an operator of convolution type but it can be interpreted as such if we look at it as a vector valued operator.

We just define the operator U with values in ℓ^2 as

$$Uf(x) = \int_{\mathbb{R}} K(x-y) f(y) dy.$$

Where K is the vector valued kernel

$$K(x) = \left\{ \frac{1}{2^n} \chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x) \right\}_n.$$

Then $\|Uf(x)\|_{\ell^2} = Sf(x)$.

The definitions and results stated above for real valued kernels remain valid for vector valued operators, if instead of absolute values one uses norms.

For example, a kernel K is in H_r if

$$\sum_{m=1}^{\infty} (2^m R)^n \quad \left(\frac{1}{(2^m R)^n} \int_{2^m R < |y| \leq 2^{m+1} R} \|K(x-y) - K(-y)\|^r \, dy \right)^{\frac{1}{r}} \leq C_r, \tag{2.12}$$

and in H_∞ if

$$\sum_{m=1}^{\infty} (2^m R)^n \sup_{2^m R < |y| \leq 2^{m+1} R} \|K(x-y) - K(-y)\| \leq C_\infty. \tag{2.13}$$

The theorem of Rubio, Ruiz and Torrea takes the form:

Theorem ([9]). *Let $1 < r \leq \infty$ and let T be a singular integral such that $K \in H_r$, then for every $0 < p < \infty$ and $w \in A_\infty$ there exists a constant C such that*

$$\int_{\mathbb{R}^n} \|Tf(x)\|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} (M_{r'} f(x))^p \, w(x) \, dx,$$

whenever the left hand side is finite.

In order to prove that our kernel is not in H_∞ we use the following property whose proof can be seen in ([10].)

Lemma 2.14. *Let us assume that*

$$0 < x < 2^i, \; 2^j < y \leq 2^{j+1},$$

where $i < j$ and $i, j \in \mathbb{Z}$.

Then $(K(x-y) - K(-y))_n = 0$ if $n \neq j$ and $(K(x-y) - K(-y))_j = \frac{\sqrt{2}}{2^j} \chi_{(2^j, x+2^j)}(y)$.

It follows that

$$\|K(x-y)-K(-y)\|_{l^2} = \frac{\sqrt{2}}{2^j}\chi_{(2^j,x+2^j)}(y).$$

From this equality it can easily be seen that the kernel K does not belong to H_∞. Indeed if $0 < x < 2^i$ and $R = 2^i$, then for any $m \in I\!N$

$$2^m 2^i \sup_{2^{m+i}<y\leq 2^{m+i+1}} \|K(x-y)-K(-y)\|_{l^2} = \frac{2^{m+i}\sqrt{2}}{2^{m+i}} = \sqrt{2}$$

and H_∞ fails.

But the same equality and Hölder's inequality give that for any $r > 1$, $K \in H_r$ and therefore:

$$\int_{\mathbb{R}} \|Uf(x)\|_{\ell^2}^p w(x)\,dx = \int_{\mathbb{R}} (Sf(x))^p w(x) \leq C \int_{\mathbb{R}} (M_{r'}f(x))^p\, w(x)\,dx,$$

This last inequality, the definition of A_p and the property of the A_p weights stated in the introduction imply that S is a bounded operator in $L^p(w)$ for any $p > 1$. For $p = 1$ one cannot deduce from the inequality above that the operator S is of weak type $(1,1)$ with respect to the measure wdx for $w \in A_1$, but the following theorem can be proved.

Theorem 2.15. *If the kernel K, of the singular integral T, is in H_r for every $\infty > r > 1$, then for any $w \in A_1$ there exists a constant C such that*

$$w\{x : |Tf(x)| > d\} \leq \frac{C}{d}\int |f(x)|w(x)dx$$

This means that in order to obtain the classical results about boundedness of the singular integrals with respect to A_p- weights one does not need property (L), not even H_∞. The intersection of the H_r is enough. (And H_∞ is a proper subset of this intersection.) This gives a partial answer to the second question.

If we want to obtain a maximal operator $M_{\mathcal{A}}$ such that $Mf(x) \leq M_{\mathcal{A}}f(x) \leq M_t f(x)$ for any f, x y $1 < t$ and

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \leq C \int_{\mathbb{R}^n} (M_{\mathcal{A}}f(x))^p\, w(x)\,dx,$$

we need to recall some basic ideas of Orlicz spaces. (See ([1])

Definition 2.16. *A function $\mathcal{B} : [0,\infty) \to [0,\infty)$ is a Young function if it is continuos, convex, increasing and satisfies $\mathcal{B}(0) = 0$, $\mathcal{B}(t) \to \infty$ when $t \to \infty$, y* $\lim_{t\to 0^+} \frac{\mathcal{B}(t)}{t} = \lim_{t\to\infty} \frac{t}{\mathcal{B}(t)} = 0$.

The Luxemburg norm of f, induced by $\mathcal{B}$, is

$$||f||_{\mathcal{B}} = \inf\left\{\lambda > 0 : \int \mathcal{B}\left(\frac{|f|}{\lambda}\right) \leq 1\right\},$$

and the $\mathcal{B}-$average of f over a cube Q es

$$||f||_{\mathcal{B},Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \mathcal{B}\left(\frac{|f|}{\lambda}\right) \leq 1\right\}.$$

By $\overline{\mathcal{B}}$ we will denote the complementary function of $\mathcal{B}$. The usual Hölder's inequality takes the form,

$$\frac{1}{|Q|}\int_Q |f\,g| \leq ||f||_{\mathcal{B},Q}||g||_{\overline{\mathcal{B}},Q}.$$

The maximal operator associated to the Young function $\mathcal{B}$ is

$$M_{\mathcal{B}}f(x) = \sup_{x\in Q} \|f\|_{\mathcal{B},Q}\ ,$$

where sup is taken over all cubes containing x. Examples of Young functions are : $B(t) = t^r$, $B(t) = e^{t^{1/k}} - 1$, $B(t) = t(1+\log^+(t))^k$. The corresponding maximal operators are M_r, $M_{\exp L^{1/k}}$ and $M_{L(1+\log^+ L)^k}$. If $k \geq 0$, $k \in \mathbb{Z}$, then $M_{L(1+\log^+ L)^k}$ is pointwise equivalent to M^{k+1},where M^k is M iterated k times. Furthermore

$$Mf(x) \leq CM_{L(1+\log^+ L)^k}f(x) \leq CM_rf(x),$$

for any $k > 0$ and $r > 1$.

Definition 2.17. *Let $\mathcal{A}$ be a Young function, we say that the kernel K satisfies the condition $L^{\mathcal{A}}$-Hörmander if there exist numbers $c_{\mathcal{A}} > 1$ y $C_{\mathcal{A}} > 0$ such that for any x and $R > c_{\mathcal{A}}|x|$,*

$$\sum_{m=1}^{\infty}(2^mR)^n||(K(x-\cdot) - K(-\cdot))\,\chi_{\{2^mR<|y|\leq 2^{m+1}R\}}(\cdot)\,||_{\mathcal{A},B(0,2^{m+1}R)} \leq C_{\mathcal{A}}.$$

The result that parallels that in ([9]) is

Theorem 2.18. *Let T be a singular integral, whose kernel K belongs to $H_{\mathcal{A}}$, then for any $0 < p < \infty$ and $w \in A_\infty$, there exists C tal que*

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C\int_{\mathbb{R}^n}(M_{\overline{\mathcal{A}}}f)^p w,$$

for any f such that the left hand side is finite.

If we want to improve inequality (1.7)we just need to find a Young function $\mathcal{A}$ such that $K \in H_{\mathcal{A}}$ and such that $M_{\overline{\mathcal{A}}}$ is between M and M_t for any $1 < t$

Lemma 2.19. *It follows from lemma(2.14 that the kernel K of the square function satisfies the condition $L^{\mathcal{A}}$-Hörmander where $\mathcal{A}(t) \approx \exp t^{\frac{1}{1+\epsilon}}$.*

If we use that the complementary function of $\mathcal{A}$ is $\mathcal{B}(t) = t(1+\log^+(t))^{1+\epsilon}$, we obtain that the inequality

$$\int_{\mathbb{R}} |Sf(x)|^p w(x)\,dx \leq C\int_{\mathbb{R}}(M_tf(x))^p\,w(x)\,dx\ t > 1$$

can be improved to

$$\int_{\mathbb{R}} |Sf(x)|^p w(x)\,dx \leq C\int_{\mathbb{R}}\left(M^3f(x)\right)^p w(x)\,dx,$$

In fact, looking at the proof more carefully we may obtain

$$\int_{\mathbb{R}} |Sf(x)|^p w(x)\,dx \leq C\int_{\mathbb{R}}\left(M^2f(x)\right)^p w(x)\,dx.$$

If we use the fact that the operator is one-sided (i.e. $Sf(x)$ depends only on the values of f at the points $y > x$,) it can be proved that

$$\int_{\mathbb{R}} |Sf(x)|^p w(x)\, dx \le C \int_{\mathbb{R}} \left((M^+)^2 f(x)\right)^p w(x)\, dx,$$

for any $w \in A_\infty^+$. which is a better estimate since it controls $\int_{\mathbb{R}} |Sf(x)|^p w(x)$ by a smaller operator and allows a wider class of weights.

3. Different weights

C. Pérez has developed a technique that allows to pass from estimates for A_∞ weights to inequalities with different weights of the type $(w, M_K w)$. Here w is arbitrary, i.e. not necessarily in A_∞ and M_K is a maximal operator that depends on the regularity properties of the kernel of the singular integral T.

Definition 3.20. *Let $\mathcal{A}$ be a Young function. We say that $\mathcal{A} \in B_p$, $p > 1$, if there exists $c > 0$ such that*

$$\int_c^\infty \frac{\mathcal{A}(t)}{t^p}\frac{dt}{t} \approx \int_c^\infty \left(\frac{t^{p'}}{\overline{\mathcal{A}}(t)}\right)^{p-1}\frac{dt}{t} < \infty.$$

The interest of this definition is that it characterizes the Young functions such that $M_{\mathcal{A}}$ is bounded in L^p (See [8]) Following his ideas one can prove:

Theorem 3.21. *Let $\mathcal{B}$ una función de Young (or $\mathcal{B}(t) = t$) and let T be a linear operator such that its adjoint, T^*, satisfies:*

$$\int_{\mathbb{R}^n} |T^* f|^q w \le C \int_{\mathbb{R}^n} (M_{\mathcal{B}} f)^q w\,,$$

for any $0 < q < \infty$ and $w \in A_\infty$. Let $1 < p < \infty$. Let us assume that there exist Young functions $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{D}$,so that $\mathcal{E} \in B_{p'}$, $\mathcal{E}^{-1}(t)\mathcal{F}^{-1}(t) \le \mathcal{B}^{-1}(t)$ and $\mathcal{D}(t) = \mathcal{F}(t^{1/p})$. Then for any weight w,

$$\int_{\mathbb{R}^n} |Tf|^p w \le C \int_{\mathbb{R}^n} |f|^p M_{\mathcal{D}} w\,.$$

Remark 3.22. *This theorem cannot ve applied to the square function S since it is not a linear operator.*

But there is a linear operator to which it can be applied. It is proved in ([2]) that for any sequence (v_n) in ℓ^∞, the operator $Tf(x) = \sum v_n(A_n - A_{n-1})f(x)$ is a, one sided, singular integral whose kernel satisfies $H_{\mathcal{A}}$ with the same $\mathcal{A}$ as the square function and in general does not satisfy H_∞. For this operator, keeping in mind that it is one sided, the preceding theorem yields:

Theorem 3.23. *For any weight w and $1 < p < \infty$*

$$\int_{\mathbb{R}} |Tf|^p w \le C \int_{\mathbb{R}} |f|^p (M^-)^{[2p]+1} w,$$

where $[t]$ is the integer part of t. and $M^- f(x) = \frac{1}{h}\sup_{h>0}\int_{x-h}^x |f|$

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