ON THE BEST RANGES FOR A_p^+ AND RH_r^+

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Abstract. In this paper we study the relationship between one-sided reverse Hölder classes RH_r^+ and the A_p^+ classes. We find the best possible range of RH_r^+ to which an A_1^+ weight belongs, in terms of the A_1^+ constant. Conversely we also find the best range of A_p^+ to which a RH_∞^+ weight belongs, in terms of the RH_∞^+ constant. Similar problems for A_p^+ , $1 < p < \infty$ and RH_r^+, $1 < r < \infty$ are solved using factorization.

1. Introduction

It is well known that there is a relationship between the A_p classes and the so called reverse Hölder classes RH_r. C. J. Neugebauer [8] has studied the following problems:

(1) For $w \in A_p$, find the precise range of $r's$ such that $w \in RH_r$, the precise range of $q < p$ for which $w \in A_q$, and the precise range of $s > 1$ so that $w^s \in A_p$.

(2) Conversely, for a fixed $w \in RH_r$, find the precise range of $p's$ such that $w \in A_p$, and the precise range of $q > r$ for which $w \in RH_q$.

For the one-sided Hardy-Littlewood maximal operator,

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|,$$

the A_p^+ classes were introduced by E. Sawyer [9]. He proved that M^+ is bounded in $L^p(w)$ ($p > 1$) if, and only if, the weight satisfies A_p^+ i.e., there exists a constant C such that for any three points $a < b < c$

$$\int_a^b w \left(\int_b^c w^{1-p'} \right)^{p-1} \leq C(c-a)^p$$

The smallest constant for which this is satisfied will be called the A_p^+ constant of w and will be denoted by $A_p^+(w)$. For $p = 1$ the weak type of the operator holds if, and only if, the weight w satisfies A_1^+ i.e. there exists C so that for any a and almost every $b > a$,

$$\int_a^b w \leq C(b-a)w(b).$$

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The smallest such constant will be called the A_1^+ constant of w and will be denoted by $A_1^+(w)$. For later reference we point out that it is an easy consequence of Lebesgue's differentiation theorem that the constant in the definiton of A_1^+ is always greater than, or equal to, one.

These classes are of interest, not only because they control the boundedness of the one-sided Hardy-Littlewood maximal operator, but they are the right classes for the weighted estimates for one-sided singular integrals [1] and they also appear in PDE [4]. In contrast to the Muckenhoupt weights, the one-sided weights are not doubling, but they satisfy a one-sided doubling property, namely if $w \in A_p^+$ then there exists C such that for any $a \in \mathbb{R}$ and $h > 0$, $\int_a^{a+2h} w \leq C \int_{a+h}^{a+2h} w$. The reverse Hölder property is not satisfied by these weights either but nevertheless Martín-Reyes [5] proved that there is a weak substitute of this notion, that we will denote by RH_r^+, which is good enough to prove the "$p-\epsilon$" property. In [7] the class A_∞^+ was introduced and it was proved that $A_\infty^+ = \cup_{p<\infty} A_p^+ = \cup_{1<r} RH_r^+$.

In this note we solve the problems of the Neugebauer paper in this context. In the proofs we will make essential use of the one-sided minimal operator introduced by Cruz-Uribe, Neugebauer and Olesen [3]. It is defined as $m^+f(x) = \inf_{c>x} \frac{1}{c-x} \int_x^c |f|$. We will also use the fact that for any positive function g, the maximal operator $M_g f(x) = \sup_{x \in I} \frac{1}{g(I)} \int_I |f| g dx$ is of weak type one-one with respect to the measure gdx. Note that for $g = 1$, we have the classical Hardy-Littlewood maximal operator, which is denoted by Mf.

The paper is organized as follows: in section 2 we give definitions and characterizations of RH_r^+, $1 < r < \infty$. In section 3 we prove two theorems of best range for the extreme classes A_1^+ and RH_∞^+. In section 4 we give a factorization theorem for weights in RH_r^+, and finally in section 5 we extend the theorems of section 3 for A_p^+ and RH_r^+, using the factorization proven in section 4. We shall see, that the index range depends on the factorization of the weight.

We end this introduction with some notation: for a given interval $I = (a, a+h)$ we denote by I^- the interval $(a-h, a)$, I^+ the interval $(a+h, a+2h)$, and I^{++} the interval $(a+2h, a+3h)$. For any $1 < p < \infty$, p' will be its conjungate exponent, if g is locally integrable and E is a measurable set, $g(E)$ will stand for $\int_E g$ and C will represent a constant that may change from time to time. Finally we remark that we can change the orientation on the real line obtaining similar results for classes RH_r^-, A_p^-, $1 < r \leq \infty$ and $1 \leq p \leq \infty$.

2. Definition, Charaterization of RH_r^+ for $1 < r < \infty$

We start this section with the definiton of RH_r^+, $1 < r < \infty$.

Definition 2.1. A weight w satisfies the one-sided reverse Hölder RH_r^+ condition, if there exists C such that for any $a < b$

$$\int_a^b w^r \leq C \left(M(w\chi_{(a,b)})(b) \right)^{(r-1)} \int_a^b w. \tag{2.2}$$

The smallest such constant will be called the RH_r^+ constant of w and will be denoted by $RH_r^+(w)$.

Definition 2.3. A weight satisfies the one-sided reverse Hölder RH_∞^+ contition, if there exists C such that

$$w(x) \le Cm^+w(x), \tag{2.4}$$

for almost all $x \in \mathbb{R}$.

The smallest such constant will be called the RH_∞^+ constant of w and will be denoted by $RH_\infty^+(w)$. It is clear that $C \ge 1$.

The following lemma gives several characterizations of RH_r^+. The constants are not necessarily the same.

Lemma 2.5. *Let $a < b < c < d$, $1 < r < \infty$, and $w \ge 0$ locally integrable, then the following staments are equivalent*

i) $\displaystyle\int_a^b w^r \le C\left(M(w\chi_{(a,b)})(b)\right)^{(r-1)}\int_a^b w.$

ii) $\displaystyle\frac{1}{b-a}\int_a^b w^r \le C\left(\frac{1}{c-b}\int_b^c w\right)^r$, *with $b-a = 2(c-b)$.*

iii) $\displaystyle\frac{1}{b-a}\int_a^b w^r \le C\left(\frac{1}{d-c}\int_c^d w\right)^r$, *with $b-a = d-b = 2(d-c)$.*

iv) $\displaystyle\frac{1}{b-a}\int_a^b w^r \le C\left(\frac{1}{c-b}\int_b^c w\right)^r$, *with $b-a = c-b$.*

v) $\displaystyle\frac{1}{b-a}\int_a^b w^r \le C\left(\frac{1}{d-c}\int_c^d w\right)^r$, *with $b-a = d-c = \gamma(d-a)$, $0 < \gamma \le \frac{1}{2}$.*

Proof. To see $i) \implies ii)$, we fix $a < b < c$, $b-a = 2(c-b)$ and take any $x \in (b,c)$. Then

$$\int_a^b w^r \le \int_a^x w^r \le C\left(M(w\chi_{(a,x)})(x)\right)^{r-1}\int_a^x w \le C\left(M(w\chi_{(a,c)})(x)\right)^{r-1}\int_a^c w.$$

Therefore $(b,c) \subset \{x : \left(M(w\chi_{(a,c)})(x)\right)^{r-1} \ge \frac{1}{C\int_a^c w}\int_a^b w^r\}$. The weak type $(1,1)$ of the Hardy-Littlewood maximal operator yields,

$$(c-b)\left(\int_a^b w^r\right)^{\frac{1}{r-1}} \le C\left(\int_a^c w\right)^{\frac{r}{r-1}},$$

which implies

$$\frac{1}{b-a}\int_a^b w^r \le C\left(\frac{1}{c-b}\int_a^c w\right)^r \le C\left(\frac{1}{c-b}\int_b^c w\right)^r,$$

the last inequality follows from the fact, proved in [7], that a weight satisfying i) satisfies A_p^+ for some p and thus it satisfies the one-sided doubling condition.

We will prove now that ii) $\implies$ i). Let us fix $a < b$ and define a sequence (x_k) as follows: $x_0 = a$ and $b - x_k = 2(b - x_{k+1})$. In particular $x_{k+1} - x_k = 2(x_{k+2} - x_{k+1}) = (b - x_{k+1})$. Using condition ii) for the points x_k, x_{k+1}, x_{k+2}, we have

$$\int_a^b w^r = \sum_0^\infty \int_{x_k}^{x_{k+1}} w^r \le C\sum_0^\infty (x_{k+1}-x_k)^{1-r}\left(\int_{x_{k+1}}^{x_{k+2}} w\right)^r$$

$$\le C\sum_0^\infty \int_{x_{k+1}}^{x_{k+2}} w\left(\frac{1}{b-x_{k+1}}\int_{x_{k+1}}^b w\right)^{r-1} \le \left(M(w\chi_{(a,b)})(b)\right)^{r-1} C\int_a^b w.$$

To see ii) $\Longrightarrow$ iii) let $a<b<c<d$ with $b-a=d-b=2(d-c)$. Using that w satisfies the one-sided doubling condition, we have

$$\frac{1}{b-a}\int_a^b w^r \le C\left(\frac{1}{c-b}\int_b^c w\right)^r \le C\left(\frac{d-c}{c-b}\frac{1}{d-c}\int_b^d w\right)^r$$
$$\le C\left(\frac{1}{d-c}\int_c^d w\right)^r.$$

iii) $\Longrightarrow$ iv) is immediate.

First of all we observe that iv) easily implies that the weight w satisfies the one-sided doubling condition. To see that iv) $\Longrightarrow$ v), let $0<\gamma\le\frac{1}{2}$ and $a<b<c<d$, $b-a=d-c=\gamma(d-a)$ then if x is the mid point between a and d we have

$$\frac{1}{b-a}\int_a^b w^r \le \frac{1}{2\gamma}\frac{1}{x-a}\int_a^x w^r \le \frac{C}{2\gamma}\left(\frac{1}{d-x}\int_x^d w\right)^r,$$

but it follows from the one-sided doubling condition that $\int_x^d w \le C_\gamma \int_c^d w$.

Suppose v) holds, let $a<b<c$, $b-a=c-b=h$, we define for $k=0,1,\dots,N$ $x_k=a+ksh$ and $y_k=b+ksh$ where $s=\frac{\gamma}{1-\gamma}$ and N is the first integer such that $(N+1)s>1$. We observe that the choice of x_k, y_k has been made so that for any $0\le k\le(N-1)$ we have $x_{k+1}-x_k=y_{k+1}-y_k=\gamma(y_{k+1}-x_k)$. Applying v), using that $r>1$ and the fact that the intervals (y_k,y_{k+1}) are disjoint, we have

$$\int_a^b w^r \le \sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}} w^r + \int_{b-sh}^b w^r$$
$$\le C(sh)^{1-r}\sum_{k=0}^{N-1}\left(\int_{y_k}^{y_{k+1}} w\right)^r + C(sh)^{1-r}\left(\int_{c-sh}^c w\right)^r$$
$$\le C_\gamma(c-a)^{1-r}\left(\int_b^c w\right)^r.$$

So we have proved that v) $\Longrightarrow$ iv).

Finally we will show that iv) $\Longrightarrow$ ii). Let $a<b<c$ with $b-a=2(c-b)$. Let x be the mid point between a,b, using the one-sided doubling property we have

$$\frac{1}{b-a}\int_a^b w^r = \frac{1}{b-a}\left(\int_a^x w^r + \int_x^b w^r\right)$$
$$= \frac{1}{2}\left(\frac{1}{x-a}\int_a^x w^r + \frac{1}{b-x}\int_x^b w^r\right)$$
$$\le \frac{C}{2}\left(\left(\frac{1}{b-x}\int_x^b w\right)^r + \left(\frac{1}{c-b}\int_b^c w\right)^r\right)$$
$$\le \frac{C}{2}\left(\left(\frac{1}{c-b}\int_x^c w\right)^r + \left(\frac{1}{c-b}\int_b^c w\right)^r\right)$$
$$\le C\left(\frac{1}{c-b}\int_b^c w\right)^r. \quad \square$$

Remark. The equivalence of i) and iv) was first proved in [3].

The following lemma tells us that in the definiton of A_p^+ we can take two intervals that are not contiguous. Note that in the case of RH_r^+ we have seen this in the previous lemma.

Lemma 2.6. *A weight $w \in A_p^+$, $p > 1$ if, and only if, there exists $0 < \gamma \leq \frac{1}{2}$ and a constant C_γ such that for any $a < b < c < d$, $b - a = d - c = \gamma(d - a)$ then*

$$\int_a^b w \left(\int_c^d w^{1-p'} \right)^{p-1} \leq C_\gamma (b-a)^p \tag{2.7}$$

Proof. If $w \in A_p^+$, $0 < \gamma \leq \frac{1}{2}$ and $a < b < c < d$, $b - a = d - c = \gamma(d - a)$ then

$$\int_a^b w \left(\int_c^d w^{1-p'} \right)^{p-1} \leq \int_a^c w \left(\int_c^d w^{1-p'} \right)^{p-1} \leq C(d-a)^p = C_\gamma (b-a)^p.$$

To prove that (2.7) implies A_p^+ we will show that (2.7) implies that for γ and a, b, c, d as above we have

$$\frac{1}{b-a} \int_a^b w \exp\left(\frac{1}{d-c} \int_c^d -\log(w) \right) \leq C.$$

Indeed

$$\begin{aligned} \frac{1}{b-a} \int_a^b w \exp\left(\frac{1}{d-c} \int_c^d -\log(w) \right) \\ = \frac{1}{b-a} \int_a^b \left[w \exp\left(\frac{1}{d-c} \int_c^d \log(w)^{1-p'} \right) \right]^{p-1} \\ \leq \frac{1}{b-a} \int_a^b w \left(\frac{1}{d-c} \int_c^d w^{1-p'} \right)^{p-1} \leq C. \end{aligned} \tag{2.8}$$

In the same way we prove that $w^{1-p'}$ satisfies

$$\exp\left(\frac{1}{b-a} \int_a^b \log(w)^{p'-1} \right) \frac{1}{d-c} \int_c^d w^{1-p'} \leq C \tag{2.9}$$

But according to part $j)$ of Theorem 1 in [7], (2.8) is equivalent to saying that $w \in A_\infty^+$, while (2.9) means that $w^{1-p'} \in A_\infty^-$ and according to Theorem 2 in [7] these two conditions imply $w \in A_p^+$

□

Remark 2.10. We can easily see that $w \in A_1^+$ if, and only if, there exists $C > 0$ such that $\frac{1}{h} \int_{a-h}^a w \leq C w(a + h)$ for almost every $a \in \mathbb{R}$ and $h > 0$.

3. The extreme cases: A_1^+ and RH_∞^+.

Theorem 3.1. *Let $w \in A_1^+$ with A_1^+ constant $C > 1$, then $w \in RH_r^+$ for any $1 < r < \frac{C}{C-1}$, and this is the best possible range.*

Proof. Let us fix the interval $I = (a,b)$. We consider the truncation of w at height N defined by $w_N = \min(w, N)$, which also satisfies A_1^+ with constant $C_N \le C$. We claim that if $\lambda_I = M(w_N\chi_I)(b)$, and $E_\lambda = \{x \in I : w_N(x) > \lambda\}$ then

$$\int_{E_\lambda} w_N \le C_N \lambda |E_\lambda| \quad \forall \lambda \ge \lambda_I. \tag{3.2}$$

Indeed if $E_\lambda = I$ we do not even need the A_1^+ condition, since

$$w_N(E_\lambda) = \int_a^b w_N \le M(w_N\chi_I)(b)(b-a) = \lambda_I(b-a) \le C_N\lambda|E_\lambda|.$$

If $E_\lambda \ne I$ we fix $\epsilon > 0$ and an open set O such that $E_\lambda \subset O \subset I$ and $|O| \le \epsilon + |E_\lambda|$. Let $J_k = (c,d)$, be one of the connected components of O. There are two cases

(1) $a \le c < d < b$,
(2) $a \le c < d = b$.

In the first case $d \notin E_\lambda$ and then $w_N(d) \le \lambda$. Now A_1^+ gives $\int_c^d w_N \le C_N w_N(d)(d-c) \le C_N\lambda(d-c)$. The second case is handled as the case $E_\lambda = I$, since $\int_c^b w_N \le M(w_N\chi_I)(b)(b-c) \le C\lambda(b-c)$. In any case $w_N(J_k) \le C_N\lambda|J_k|$. Adding up we get

$$w_N(E_\lambda) \le w_N(O) \le C_N\lambda|O| \le C_N\lambda(\epsilon + |E_\lambda|).$$

Since ϵ was arbitrary we are done. Now we proceed in the standard way i.e., we fix $s > -1$, multiply both sides of (3.2) by λ^s and integrate from λ_I to infinity to obtain,

$$\frac{1}{s+1}\int_I (w_N^{s+2} - \lambda_I^{s+1} w_N) \le \frac{C_N}{s+2}\int_I w_N^{s+2}.$$

Now if $r = s+2 < \frac{C_N}{C_N-1}$ then $\frac{1}{s+1} - \frac{C_N}{s+2} > 0$, and we get

$$\int w_N^r \le C_N \lambda_I^{r-1}\int_I w_N = C_N(M(w_N\chi_I)(b))^{r-1}\int_I w_N.$$

Now $C_N \le C$ implies $\frac{C_N}{C_N-1} \ge \frac{C}{C-1}$, and therefore if $r \le \frac{C}{C-1}$ then

$$\int_a^b w_N^r \le C_N(M(w_N\chi_{(a,b)})(b))^{r-1}\int_a^b w_N \le C(M(w\chi_{(a,b)})(b))^{r-1}\int_a^b w$$

and the monotone convergence theorem gives $w \in RH_r^+$. To see that this is the best possible range we consider the function

$$w(x) = x^{\frac{1}{C}-1}\chi_{(0,\infty)}(x).$$

It is clear that does not satisfy $RH^+_{\frac{C}{C-1}}$ because $w^{\frac{C}{C-1}}(x) = \frac{1}{x}$ for $x > 0$. To see that it satisfies A_1^+ with constant C, we consider three cases

(1) $a < b \le 0$
(2) $a \le 0 < b$
(3) $0 < a < b$

In the first case there is nothing to check. In the second case $\frac{1}{b-a}\int_a^b w < \frac{1}{b}\int_0^b w(x) = \frac{C}{b}b^{\frac{1}{C}} = Cw(b)$. Finally if $0<a<b$, $\int_a^b w = C(b^{\frac{1}{C}} - a^{\frac{1}{C}}) \le C(b-a)w(b)$ $\square$

Remark. Note that if $C=1$, then $w(x) = M^- w(x)$, and this implies that w is non-decreasing. This tells us that $w \in RH_\infty^+$.

Theorem 3.3. *If w satisfies RH_∞^+ with constant $C>1$, then $w \in A_p^+$ for all $p > C$, and this is the best possible range.*

Proof. A truncation argument as in Theorem 3.1 allows us to suppose that w is bounded away from zero, i.e. there exists $\beta > 0$ so that $w(x) \ge \beta$ for all x. Let us fix $I=(a,b)$ and consider $\lambda_I = m^+(w\frac{1}{\chi_I})(a)$. We claim that if $\lambda < \lambda_I$ and $E_\lambda = \{x \in I : w(x) < \lambda\}$, then

$$\lambda |E_\lambda| \le C\int_{E_\lambda} w. \tag{3.4}$$

As before if $E_\lambda = I$ then $\lambda|E_\lambda| = \lambda(b-a) < \lambda_I(b-a) = \int_a^b w \le w(E_\lambda)$. If $E_\lambda \ne I$ then we aproximate it by an open set $O = \cup J_k$ where $E_\lambda \subset O \subset I$ and $w(O) < \epsilon + w(E_\lambda)$. Let us fix $J_k = (c,d)$. There are two cases

(1) $a < c$
(2) $a = c$.

In the first case $c \notin E_\lambda$ and then $\lambda(d-c) \le w(c)(d-c) \le Cm^+w(c)(d-c) \le C\int_c^d w$. In the second case $\lambda(d-c) \le \lambda_I(d-a) \le \int_a^d w$, and (3.4) follows. If we multiply both sides of (3.4) by λ^{-r} with $r>2$ and integrate we have

$$\int_0^{\lambda_I} \lambda^{1-r} \int \chi_{E_\lambda}(x)\,dx\,d\lambda \le C\int_0^\infty \lambda^{-r}\int_{E_\lambda} w(x)\,dx\,d\lambda.$$

For the left hand side we obtain,

$$\begin{aligned}\int_\beta^{\lambda_I} \lambda^{1-r}\int \chi_{E_\lambda}(x)\,dx\,d\lambda &= \frac{1}{2-r}\int_{\{x\in I:\, w(x)<\lambda_I\}} \lambda_I^{2-r} - w^{2-r}\,dx \\ &\ge \frac{1}{2-r}\int_I \lambda_I^{2-r} - w^{2-r}\,dx = \frac{1}{r-2}\int_I w^{2-r} - \frac{|I|}{r-2}\lambda_I^{2-r},\end{aligned}$$

while the right hand side is equal to $\frac{C}{r-1}\int_I w^{2-r}$. Therefore

$$\frac{1}{r-2}\int_I w^{2-r} \le \frac{C}{r-1}\int_I w^{2-r} + \frac{|I|}{r-2}\lambda_I^{2-r}.$$

If we choose $r>2$ such that $C(r-2) < (r-1)$, we obtain that there exists C so that

$$\frac{1}{|I|}\int_I w^{2-r} \le C\left(m^+\left(\frac{w}{\chi_I}\right)(a)\right)^{2-r}. \tag{3.5}$$

We now claim that (3.5) implies that $w \in A_p^+$ with $p = \frac{r-1}{r-2}$. Let us fix $a<b<c$ and choose $x \in (a,b)$. If we keep in mind that $1-p' = 2-r$ we may write

$$\left(\frac{1}{c-a}\int_b^c w^{1-p'}\right)^{p-1} \le \left(\frac{1}{c-x}\int_x^c w^{1-p'}\right)^{p-1} \le C\left(m^+(\frac{w}{\chi_{(x,c)}})(x)\right)^{-1},$$

but

$$\left(m^+(\frac{w}{\chi_{(x,c)}})(x)\right)^{-1} = \left(\inf_{x<d<c} \frac{1}{d-x}\int_x^d w\right)^{-1} = \sup_{x<d<c} \frac{d-x}{\int_x^d w} = M_w\left(\frac{\chi_{(a,c)}}{w}\right)(x).$$

We have thus proved that if $\lambda = \left(\frac{1}{c-a}\int_b^c w^{1-p'}\right)^{p-1}$ then

$$(a,b) \subset \{x : CM_w\left(\frac{\chi_{(a,c)}}{w}\right)(x) > \lambda\},$$

and the weak type of M_w with respect to the measure wdx yields $\int_a^b w \leq C(c-a)^p \left(\int_b^c w^{1-p'}\right)^{1-p}$ which is A_p^+. Finally it can be checked that the function $w(x)$ which is 0 for $x < -1$, identically one for $x > 0$ and $|x|^{C-1}$ between -1 and 0, satisfies RH_∞^+ with constant C, but is not in A_C^+. □

Remark. Note that if $C = 1$, then $w(x) = m^+w(x)$, and this implies that w is non-decreasing. This tells us that $w \in A_1^+$.

We had several different characterizations of RH_r^+, one involved the maximal operator, but dealt with one interval, and the others involved two intervals but no operator. We can now prove that for RH_∞^+ the situation is the same, we can characterize RH_∞^+ using two intervals instead of the minimal operator.

Corollary 3.6. *$w \in RH_\infty^+$ if, and only if, there exists C such that for any interval I,*

$$esssup_I w \leq C\frac{1}{|I^+|}\int_{I^+} w \tag{3.7}$$

Proof. It is immediate that (3.7) implies RH_∞^+. Assume now that $w \in RH_\infty^+$. The preceding theorem tells us that $w \in A_p$ for some p, and therefore it satisfies the one-sided doubling condition. Therefore if $I = (a,b)$ is any interval, $I^+ = (b,c)$ and $x \in I$ we have

$$w(x) \leq \frac{C}{c-x}\int_x^c w \leq \frac{C}{c-b}\int_b^c w,$$

which is (3.7). □

Remark. Note that with this definition, we have that $RH_\infty^+ \subset \cap_{r>1} RH_r^+$.

4. Factorization of weights in RH_r^+, $1 < r \leq \infty$.

The theorems on the best range for weights in A_p^+ ($p > 1$) or in RH_r^+, $r < \infty$ will be stated in terms of factorizations of the given weight. Therefore this section will be devoted to prove a factorization of functions in RH_r^+. The bilateral case was studied in [2].

Definition 4.1. A function w is said to be essentially increasing if there exists C so that $w(x) \leq Cw(y)$ for any $x < y$.

Lemma 4.2. *A function belongs to $RH_\infty^+ \cap A_1^+$ if, and only if, it is essentially increasing.*

Proof. Assume that $w \in RH_\infty^+ \cap A_1^+$ and $x < y$ then $w(x) \leq C\frac{1}{y-x}\int_x^y w \leq Cw(y)$ and w is essentially increasing. Conversely, if w is essentially increasing then for any x and $h > 0$ we have $w(x) \leq \frac{C}{h}\int_x^{x+h} w$, then $w \in RH_\infty^+$. On the other hand $\frac{1}{h}\int_{x-h}^x w \leq Cw(x)$, so $w \in A_1^+$ □

Lemma 4.3. *Let $1 < r \leq \infty$ and $1 \leq p < \infty$.*

(1) *If u is essentially increasing and $v \in RH_r^+$ then $uv \in RH_r^+$.*
(2) *If u is essentially increasing and $v \in A_p^+$ then $uv \in A_p^+$.*

Proof. This proof follows immediately from Definition 4.1. □

Lemma 4.4. *Let $1 < r \leq \infty$ and $1 \leq p < \infty$. $w \in RH_r^+ \cap A_p^+$ if, and only if, $w^r \in A_q^+$, with $q = r(p-1)+1$.*

Proof. Let $C_1 = RH_r^+(w)$, and $C_2 = A_p^+(w)$, $w \in RH_r^+ \cap A_p^+$, and $q = r(p-1)+1$. Also note that $1 - q' = 1 - \frac{r(p-1)+1}{r(p-1)} = \frac{1}{r(1-p)}$,

$$\left(\frac{1}{|I^-|}\int_{I^-} w^r\right)\left(\frac{1}{|I^+|}\int_{I^+} w^{r(1-q')}\right)^{q-1}$$
$$\leq C_1\left(\frac{1}{|I|}\int_I w\right)^r\left(\frac{1}{|I^+|}\int_{I^+} w^{1-p'}\right)^{r(p-1)}$$
$$\leq C_1 C_2^r,$$

and by Lemma 2.6 we have that $w^r \in A_q^+$.

If $w^r \in A_q^+$, by Hölder's inequality

$$\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I^+|}\int_{I^+} w^{-1/(p-1)}\right)^{p-1}$$
$$\leq \left(\frac{1}{|I|}\int_I w^r\right)^{1/r}\left(\frac{1}{|I^+|}\int_{I^+} w^{-r/(q-1)}\right)^{(q-1)/r}$$
$$\leq C^{1/r},$$

obtaining in this way that $w \in A_p^+$. Now again by Hölder's inequality

$$1 = \frac{1}{|I^+|}\int_{I^+} w^{-1/p}w^{1/p} \leq \left(\frac{1}{|I^+|}\int_{I^+} w\right)^{1/p}\left(\frac{1}{|I^+|}\int_{I^+} w^{-p'/p}\right)^{1/p'}$$

so

$$\left(\frac{1}{|I^+|}\int_{I^+} w^{-1/(p-1)}\right)^{1-p} \leq \frac{1}{|I^+|}\int_{I^+} w,$$

and we get

$$\left(\frac{1}{|I|}\int_I w^r\right)^{1/r} \leq C\left(\frac{1}{|I^+|}\int_{I^+} w^{-r/(q-1)}\right)^{-(q-1)/r} = C\left(\frac{1}{|I^+|}\int_{I^+} w^{-1/(p-1)}\right)^{1-p}$$
$$\leq C\frac{1}{|I^+|}\int_{I^+} w,$$

proving that $w \in RH_r^+$. □

Factorization Theorem for weights in $RH_r^+ \cap A_p^+$. *A weight $w \in RH_r^+ \cap A_p^+$ with $1 \le p < \infty$, $1 < r \le \infty$ if, and only if, there exists weights w_0 and w_1 such that $w_0 \in RH_r^+ \cap A_1^+$, $w_1 \in RH_\infty^+ \cap A_p^+$ and $w = w_0 w_1$.*

Observe that since $\cup_{p<\infty} A_p^+ = \cap_{1<r} RH_r^*$ every weight in RH_r^+ is in some A_p^+ . See [7].

Proof. Let us consider first the cases $p = 1$ or $r = \infty$.
If $p = 1$ and $r \le \infty$, we put $w_1 = 1$, and $w_0 = w$, then obviously $w_0 \in RH_r^+ \cap A_1^+$, and $w_1 \in RH_\infty^+ \cap A_1^+$.
If $p \ge 1$ and $r = \infty$, we put $w_0 = 1$, and $w_1 = w$, obtaining $w_0 \in RH_\infty^+ \cap A_1^+$, $w_1 \in RH_\infty^+ \cap A_p^+$.
Conversely, given w_0 and w_1, at least one of them belongs to $RH_\infty^+ \cap A_1^+$, (because $p = 1$ or $r = \infty$), so one of them is essentially increasing, therefore $w_0 w_1 \in RH_r^+ \cap A_p^+$ (Lemma 4.3).
Let us suppose now, $p > 1$ and $r < \infty$. Let $w = w_0 w_1$, with $w_0 \in RH_r^+ \cap A_1^+$, and $w_1 \in RH_\infty^+ \cap A_p^+$, we want to see that $w \in RH_r^+ \cap A_p^+$. Note that for w_1 the following holds

$$\frac{1}{|I|}\int_I w_1^{1-p'} \le C\left(\frac{1}{|I^-|}\int_{I^-} w_1\right)^{1-p'} \le C w_1(a-h)^{1-p'},$$

this implies, $w_1^{1-p'} \in A_1^-$ (Remark 2.10). Let $v = w_1^{1-p'}$, then $w_1 = v^{1-p}$ with $v \in A_1^-$, so $w = w_0 w_1 = w_0 v^{1-p}$ with $w_0 \in A_1^+$ and $v \in A_1^-$ (see [7]), and this implies $w \in A_p^+$.
Now

$$\begin{aligned}\frac{1}{|I|}\int_I w^r = \frac{1}{|I|}\int_I w_0^r w_1^r &\le (\sup_I w_1)^r C\left(\frac{1}{|I^+|}\int_{I^+} w_0\right)^r \\ &\le C\left(\frac{1}{|I^+|}\int_{I^{++}} w_1\right)^r \left(\inf_{I^{++}} w_0\right)^r \\ &\le C\left(\frac{1}{|I^{++}|}\int_{I^{++}} w_0 w_1\right)^r,\end{aligned}$$

by Lemma 2.5, we have $w \in RH_r^+$. Conversely let $w \in RH_r^+ \cap A_p^+$, then by Lemma 4.4 $w^r \in A_q^+$, with $q = r(p-1)+1$, there exists $v_0 \in A_1^+$, and $v_1 \in A_1^-$, such that $w^r = v_0 v_1^{1-q}$ (see [7]), or equivalently $w = v_0^{1/r} v_1^{(1-q)/r} = v_0^{1/r} v_1^{1-p}$. Let $w_0 = v_0^{1/r}$ and $w_1 = v_1^{1-p}$. We will see that $w_0 \in RH_r^+ \cap A_1^+$. We note,

$$\begin{aligned}\frac{1}{|I|}\int_I w_0^r = \frac{1}{|I|}\int_I v_0 &\le C \inf_{I^+} v_0 \\ &\le C\left(\frac{1}{|I^+|}\int_{I^+} v_0^{1/r}\right)^r = C\left(\frac{1}{|I^+|}\int_{I^+} w_0\right)^r,\end{aligned}$$

and also,

$$\begin{aligned}\frac{1}{|I|}\int_I w_0 = \frac{1}{|I|}\int_I v_0^{1/r} &\le \left(\frac{1}{|I|}\int_I v_0\right)^{1/r} \\ &\le C \inf_{I^+} v_0^{1/r} = C \inf_{I^+} w_0.\end{aligned}$$

We only have to see now, that $w_1 \in RH_\infty^+ \cap A_p^+$ and we are done.

First we claim

$$w \in A_1^- \text{ then } w^{-\gamma} \in RH_\infty^+, \text{ for all } \gamma > 0. \tag{4.5}$$

In fact by Hölder's inequality, we have for any interval $I = (a,b)$, $\left(\frac{1}{|I|}\int_I w\right)^{-\gamma} \leq \frac{1}{|I|}\int_I w^{-\gamma}$ and as $w \in A_1^-$ we have that for almost every $x \in I^-$, $Cw(x) \geq \frac{1}{|I|}\int_I w$, and therefore

$$w(x)^{-\gamma} \leq C\left(\frac{1}{|I|}\int_I w\right)^{-\gamma} \leq \frac{1}{|I|}\int_I w^{-\gamma} \leq C\frac{1}{b-x}\int_x^b w^{-\gamma}.$$

Let $w_1 = v_1^{1-p}$. As $v_1 \in A_1^-$, then $w_1 \in RH_\infty^+$. Moreover

$$\begin{aligned}
\frac{1}{|I|}\int_I w_1 \left(\frac{1}{|I^+|}\int_{I^+} w_1^{1-p'}\right)^{p-1} &= \frac{1}{|I|}\int_I v_1^{1-p}\left(\frac{1}{|I^+|}\int_{I^+} v_1\right)^{p-1} \\
&\leq \frac{1}{|I|}\int_I v_1^{1-p}(C\inf_I v_1)^{p-1} \\
&\leq \frac{C}{|I|}\int_I v_1^{1-p}v_1^{p-1} \leq C,
\end{aligned}$$

i.e. $w_1 \in A_p^+$. □

Factorization Theorem for weights in A_∞^+. *A weight $w \in A_\infty^+$ if, and only if, there exists $w_1 \in RH_\infty^+$ and $w_0 \in A_1^+$ such that $w = w_0 w_1$.*

Proof. If $w \in A_\infty^+$ then $w \in A_q^+$ for some $1 < q < \infty$, so there exist $v_0 \in A_1^+$ and $v_1 \in A_1^-$ such that $w = v_0 v_1^{1-q}$. Let $w_0 = v_0$ and $w_1 = v_1^{1-q}$. By (4.5) $w_1 \in RH_\infty^+$. So we are done. Conversely if $w_1 \in RH_\infty^+$, then $w_1 \in A_q^+$ for some $1 < q$, i.e., there exists C such that

$$\left(\frac{1}{|I|}\int_I w_1\right)^{q'-1}\frac{1}{|I|}\int_{I^+} w_1^{1-q'} \leq C,$$

but then

$$(\sup_{I^-} w_1)^{q'-1}\frac{1}{|I|}\int_{I^+} w_1^{1-q'} \leq \left(\frac{1}{|I|}\int_I w_1\right)^{q'-1}\frac{1}{|I|}\int_{I^+} w_1^{1-q'} \leq C,$$

and we get

$$\frac{1}{|I|}\int_{I^+} w_1^{1-q'} \leq C\inf_{I^-} w_1^{1-q'},$$

and it is easy to see that this inequality implies $w_1^{1-q'} \in A_1^-$. Then $v_1 = w_1^{1-q'} \in A_1^-$, so $w = w_0 w_1 = w_0 v_1^{1-q} \in A_q^+ \subset A_\infty^+$. □

5. Classes A_p^+ and RH_r^+.

In this section we will use Theorems 3.1 and 3.3 and the factorization theorems to obtain the best ranges for the classes A_p^+ and RH_r^+. As we shall see, the range of the index will depend on the factorization of the weights.

The following theorem gives us the precise range in A_p^+ for weights in RH_r^+.

Theorem 5.1. *Let $w \in RH_r^+$, $w = w_0 w_1^{\frac{1}{r}}$ with $w_0 \in RH_\infty^+$ and $w_1 \in A_1^+$, then $w \in A_p^+$ for all $p > C$, where $C = RH_\infty^+(w_0)$ and this is the best possible range.*

Proof. Let $w_0 \in RH_\infty^+$ and $w_1 \in A_1^+$. By Theorem 3.3 $w_0 \in A_p^+$ for all $p > C$. Let $p > C$, there exists $\epsilon > 0$ such that $w_0 \in A_{p-\epsilon}^+$, so we choose $s > 1$ satisfying $1-(p-\epsilon)' = s(1-p')$, and by Hölder's inequality

$$\frac{1}{|I^-|}\int_{I^-} w_0 w_1 \left(\frac{1}{|I^+|}\int_{I^+}(w_0w_1)^{1-p'}\right)^{p-1} \leq$$

$$\left(\frac{1}{|I|}\int_I w_0\right)\left(\frac{1}{|I^-|}\int_{I^-} w_1\right)\left(\frac{1}{|I^+|}\int_{I^+} w_0^{s(1-p')}\right)^{\frac{(p-1)}{s}}\left(\frac{1}{|I^+|}\int_{I^+} w_1^{s'(1-p')}\right)^{\frac{(p-1)}{s'}} \leq$$

C.

To see that this is the best range, we consider w_0 as in Theorem 3.3 and $w_1 = 1$. □

Remark 5.2. Given $w \in RH_r^+$ there exist $u \in RH_\infty^+$, and $v \in A_1^+$ such that $w = uv^{\frac{1}{r}}$. We only have to consider the factorization theorem and choose $u = w_1$ and $v = w_0^r$. We have to prove that $v \in A_1^+$. Keeping in mind that $w_0 \in RH_r^+ \cap A_1^+$ we have

$$\frac{1}{|I^-|}\int_{|I^-|} v = \frac{1}{|I^-|}\int_{|I^-|} w_0^r \leq C\left(\frac{1}{|I|}\int_{|I|} w_0\right)^r \leq Cw_0^r(x) = Cv(x),$$

for almost every $x \in I^+$, i.e., $v \in A_1^+$.

The next theorem shows us the precise range of the higher integrability of $w \in RH_r^+$.

Theorem 5.3. *Let $w \in RH_r^+$, $w = uv^{1/r}$ with $u \in RH_\infty^+$ and $v \in A_1^+$. If $C = A_1^+(v)$ then $w \in RH_s^+$ for all $r \leq s < \frac{Cr}{C-1}$. The range of s is the best possible.*

Proof. Let $r < s < \frac{Cr}{C-1}$, we choose $q > 1$ such that $s < \frac{Cr}{q(C-1)}$. As $1 < \frac{qs}{r} < \frac{C}{C-1}$, by Theorem 3.1 $v \in RH_{\frac{qs}{r}}^+$, using Hölder's inequality, that $u^s \in RH_\infty^+$ and $v \in A_1^+$ we have,

$$\begin{aligned}
\frac{1}{|I|}\int_I w^s &= \frac{1}{|I|}\int_I u^s v^{s/r} \leq \left(\frac{1}{|I|}\int_I u^{q's}\right)^{1/q'}\left(\frac{1}{|I|}\int_I v^{qs/r}\right)^{1/q} \\
&\leq \sup_I u^s C\left(\frac{1}{|I^+|}\int_{I^+} v\right)^{s/r} \leq \frac{C}{|I^+|}\int_{I^+} u^s\left(\inf_{I^{++}} v\right)^{s/r} \\
&\leq C\sup_{I^+} u^s \inf_{I^{++}} v^{s/r} \leq C\left(\frac{1}{|I^{++}|}\int_{I^{++}} u\right)^s \inf_{I^{++}} v^{s/r} \\
&\leq C\left(\frac{1}{|I^{++}|}\int_{I^{++}} uv^{1/r}\right)^s = C\left(\frac{1}{|I^{++}|}\int_{I^{++}} w\right)^s,
\end{aligned}$$

and we get that $w \in RH_s^+$, (Lemma 2.5).
To see this is the best range possible, we choose $v \in A_1^+$ as in Theorem 3.1 and $u = 1$, then $w = v^{1/r} \in RH_s^+$ for all $r \leq s < \frac{Cr}{C-1}$ ($C = A_1^+(v)$). If $s = \frac{Cr}{C-1}$ and $w \in RH_s^+$ then $v \in RH^+_{\frac{C}{C-1}}$, but we have seen (Theorem 3.1) that this can not happen. □

The next theorem shows us which is the best range in RH_r^+ for a given weight in A_p^+.

Theorem 5.4. *Let $w \in A_p^+$, $w = uv^{1-p}$, with $u \in A_1^+$, $v \in A_1^-$ and $C = A_1^+(u)$, then $w \in RH_r^+$ for all $1 < r < \frac{C}{C-1}$, being this range the best possible.*

Proof. By Theorem 3.1 $u \in RH_r^+$ for all $1 < r < \frac{C}{C-1}$ and we know that $v^{1-p} \in RH_\infty^+$, then

$$\begin{aligned}
\frac{1}{|I|}\int_I w^r &\leq \frac{1}{|I|}\int_I u^r \sup_I \left(v^{-r(p-1)}\right) \\
&\leq C\left(\frac{1}{|I^+|}\int_{I^+} u\right)^r \left(\frac{1}{|I^+|}\int_{I^+} v^{1-p}\right)^r \leq C\left(\inf_{I^{++}} u\right)^r \left(\sup_{I^+} v^{1-p}\right)^r \\
&\leq C\left(\inf_{I^{++}} u\right)^r \left(\frac{1}{|I^{++}|}\int_{I^{++}} v^{1-p}\right)^r \leq C\left(\frac{1}{|I^{++}|}\int_{I^{++}} w\right)^r.
\end{aligned}$$

By Lemma 2.5 $w \in RH_r^+$.
To see this is the best range we take u as in Theorem 3.1 and $v = 1$. So we have $w = u \in A_p^+$, and $w \notin RH^+_{\frac{C}{C-1}}$. □

Corollary 5.5. *Let $w = uv^{1-p} \in A_p^+$ with $u \in A_1^+$, $v \in A_1^-$ and $C = \max\{A_1^+(u), A_1^-(v)\}$, then $w^\tau \in A_p^+$ for all $1 \leq \tau < \frac{C}{C-1}$ and the range is the best possible.*

Proof. By Theorem 5.4 we have that $w \in RH_\tau^+$ for all $1 \leq \tau < \frac{C}{C-1}$ and $w^{1-p'} \in RH_\tau^-$ for all $1 \leq \tau < \frac{C}{C-1}$. Let $a < d$, we choose b, c such that $b-a = d-c = \frac{1}{4}(d-a)$, and we also choose the point $\frac{c+b}{2}$. Then, we have four intervals, namely, $I^- = (a, b)$, $I = (b, \frac{b+c}{2})$, $I^+ = (\frac{b+c}{2}, c)$, and $I^{++} = (c, d)$. Now

$$\begin{aligned}
\frac{1}{|I^-|}\int_{I^-} w^\tau \left(\frac{1}{|I^{++}|}\int_{I^{++}} w^{\tau(1-p')}\right)^{p-1} &\leq \left(\frac{1}{|I|}\int_I w\right)^\tau \left(\frac{1}{|I^+|}\int_{I^+} w^{1-p'}\right)^{\tau(p-1)} \\
&\leq C^\tau,
\end{aligned}$$

thus $w^\tau \in A_p^+$, (Lemma 2.6). Considering u as in Theorem 3.1, we see this is the best possible range. □

Using Theorem 5.4 we will show the exact range of $q < p$ such that $w \in A_p^+$ implies $w \in A_q^+$.

Theorem 5.6. *Let $w = uv^{1-p} \in A_p^+$ with $u \in A_1^+$, $v \in A_1^-$ and $C = A_1^-(v)$, then $w \in A_q^+$ for all $1 + \frac{(p-1)(C-1)}{C} < q < \infty$ and this is the best range for q.*

Proof. Note that $w^{1-p'} = vu^{1-p'} \in A_{p'}^-$, by Theorem 5.4 $w^{1-p'} \in RH_r^-$ for all $1 < r < \frac{C}{C-1}$. From lemma 4.4 for the classes RH_r^- and A_p^- we have that $w^{(1-p')r} \in A_{q'}^-$

where $q' = r(p'-1)+1 = \frac{r}{p-1}+1$. But this is the same as $w^{1-q'} \in A^-_{q'}$ i.e., $w \in A^+_q$ for all $1+(p-1)\frac{C-1}{C} < q$.
To see this is the best range, let $v(x) = x^{\frac{1-C}{C}}$ if $x \le 0$ and equal to 0 if $x > 0$ and $u = 1$ for all x. Note that $v \in A^-_1$ and $A^-_1(v) = C$. Then $w = v^{1-p} \in A^+_p$ and $w \in A^+_q$ for all $q > 1+(p-1)\frac{C-1}{C}$. Observe that $w \notin A^+_{1+(p-1)\frac{C-1}{C}}$. □

Finally the last theorem gives us the best possible range, for a weight in A^+_∞.

Theorem 5.7. *Let $w \in A^+_\infty$, $w = w_0 w_1$, $w_0 \in A^+_1$, $w_1 \in RH^+_\infty$ and $C = RH^+_\infty(w_1)$, then $w \in A^+_p$ for all $p > C$. The range of $p's$ is the best possible.*

Proof. Note that $w_1 \in RH^+_\infty$ implies $w_1 \in A^+_p$ for all $p > C$, then

$$\begin{aligned}
&\frac{1}{|I|}\int_I w_0 w_1 \left(\frac{1}{|I|^{++}}\int_{I^{++}} (w_0 w_1)^{1-p'}\right)^{p-1} \\
&\quad \le \sup_I (w_1)\,\frac{1}{|I|}\int_I w_0 \left(\sup_{I^{++}} w_0^{1-p'}\right)^{p-1} \left(\frac{1}{|I|^{++}}\int_{I^{++}} w_1^{1-p'}\right)^{p-1} \\
&\quad \le C\frac{1}{|I|^{+}}\int_{I^+} w_1 \,\inf_{I^+}(w_0)\,\sup_{I^{++}}(w_0^{-1}) \left(\frac{1}{|I|^{++}}\int_{I^{++}} w_1^{1-p'}\right)^{p-1} \\
&\quad \le C\,\sup_{I^{++}}(w_0^{-1})\,\frac{1}{|I^+|}\int_{I^+} w_0 \\
&\quad \le C\,\frac{1}{(\inf_{I^{++}} w_0)}\,\inf_{I^{++}} w_0 \le C,
\end{aligned}$$

by Lemma 2.6 $w \in A^+_p$ for all $p > C$.
To see this is the best range, we consider $w(x) = 0$ if $x \le -1$, $|x|^{C-1}$ if $-1 < x \le 0$ and 1 if $x \ge 0$ □

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