A DISSIPATION-BASED ALGORITHM WITH ENERGY CONTROL FOR GEOMETRICALLY NONLINEAR ELASTODYNAMICS USING EIGHT-NODE FINITE ELEMENTS AND ONE-POINT QUADRATURE

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Abstract. A formulation for the geometrically nonlinear dynamic analysis of elastic structures is presented in this paper. It is well known that the Newmark’s method, which is considered the most popular time-stepping scheme for structural dynamics, exhibits unconditional stability in the case of linear dynamical systems. However, this characteristic is lost in the nonlinear regime owing to the lack of an energy balance within each time step of the integration process. In order to obtain a numerical scheme with energy-conserving and controllable numerical dissipation properties a new algorithm is proposed in this work. The formulation is based on the Generalized-α method, adjusting optimized time integration parameters and the addition of an algorithmic control of the energy balance restriction, which is introduced in the Newton-Raphson iterative process within each time step of the time marching. The Finite Element Method (FEM) is employed in the present model for spatial discretizations using an eight-node hexahedral isoparametric element with one-point quadrature. In order to avoid the excitation of spurious modes an efficient hourglass control technique is used and therefore volumetric locking as well as shear locking are not observed. Some examples are analyzed in order to validate the present algorithm.
1 INTRODUCTION

The development of numerical algorithms to simulate the dynamic response of highly nonlinear elastic structures is a major topic in the field of Elastodynamics. In the Finite Element Method (FEM) framework (see Zienkiewicz and Taylor, 2005) the discretized structural equation of motion needs to be temporally integrated using some of the existing time-stepping schemes and they must be chosen between explicit and implicit algorithms.

In practice, explicit algorithms present a severe limitation (specially if the problem is governed by low frequencies) related to the time step size employed in the time integration, which must be small enough to describe correctly the highest frequency of the discrete system and maintain the numerical stability. On the other hand, for implicit algorithms the time step may be selected independent of stability considerations leading to great savings on computational time, although they require more computer memory. In addition, unconditionally stable schemes must possess some form of numerical dissipation in order to damp out spurious participation of higher modes on the structural response.

Examples of one-step algorithms commonly used in structural dynamics, which are based on unconditionally stable numerical integration, are the Houbolt’s method, the Wilson $\theta$-method and the Newmark’s method (see Bathe, 1996). The Wilson method is generally too dissipative in the lower modes, requiring the time step employed in the time integration to be smaller than that needed for accuracy. The Houbolt’s method is known to be even more dissipative than the Wilson method. Moreover, it does not permit parametric control over the amount of numerical dissipation. Finally, the Newmark method allows the numerical dissipation to be continuously controlled by a user parameter instead of the time step. However, the dissipation properties are considered to be inferior to the previously mentioned methods, leading to low-frequency damping and reduction to first-order accuracy. This drawback motivated a sequence of papers in which the algorithmic damping characteristic was modified by satisfying the equation of motion in an average sense within each time interval (see Hilber et al., 1977, Wood et al., 1981 and Chung and Hulbert, 1993). These generalized methods reached great popularity due to their simplicity of implementation for both linear and nonlinear problems. Unfortunately, the energy conservation properties of the dynamic system were not verified.

Owing to its effectiveness, the Newmark method is probably the most widely used time discretization scheme in nonlinear structural analysis. However, the unconditional stability observed for the linear regime is frequently lost when the method is applied to nonlinear cases. The reason for this failure is related to the lack of algorithmic conservation or decrease of the total energy within each time step, which is a sufficient energy criterion for nonlinear dynamics. This conclusion was described first by Belytschko and Schoeberle (1975) and pointed out recently by Kuhl and Crisfield (1999).

Simo and Tarnow (1992) are recognized as one of the first authors to conceive an algorithm, denoted by Energy-Momentum method, which guarantee unconditional stability in nonlinear dynamics of three-dimensional elastic bodies. The main contribution of this work is the discovery that the key for an algorithmic implementation of a time-stepping procedure, which ensures unconditional stability, is on the treatment of the stress update. A mean-value theorem argument is used to show that a solution obeying an energy balance restriction is guaranteed to exist (see Simo and Tarnow, 1992). A lot of works have followed the Simo-Tarnow theorem in order to develop new algorithms based on finite element models for finite deformation and rotations, as for example, Crisfield and Shi (1994), Kuhl and Ramm (1996), Kuhl and Crisfield (1999) and Laursen and Meng (2001).

In this paper a modification of the numerical scheme presented by Kuhl and Crisfield...
(1999) for dynamic analysis of nonlinear elastic structures is proposed, which is performed with the addition of an algorithmic control of the energy balance restriction by means of a fictitious viscous term incorporated into the Newton-Raphson iterative process within each time interval. The scheme employed by Kuhl and Crisfield (1999) includes the Generalized-α method developed by Chung and Hulbert (1993) and optimized algorithmic parameters that lead to a time integration process characterized by low numerical dissipation of the lower frequencies and high numerical dissipation of the higher ones. Moreover, the idea of Simo-Tarnow’s theorem is transferred into the Generalized-α method by substituting the internal force vector and the tangential stiffness matrix by the weighted average of these variables at the beginning and the end of the time step. Another aspect to be investigated here is the behaviour of an eight-node hexahedral finite element with one-point quadrature for highly nonlinear dynamic problems. A formulation for geometrically nonlinear analysis is presented according to the model proposed by Duarte Filho and Awruch (2004). Volumetric locking is eliminated by a uniform reduced integration of the gradient matrix (reminding the reduced selective integration process, which was intensively used to prevent volumetric locking; see, for instance, Hughes, 1980) and shear locking is avoided by describing the strain rate tensor in a co-rotational coordinates system. The present formulation is restricted to the particular case of small strains and large displacements and rotations, where the constitutive equation is given according to the hypoelastic description. The Truesdell rate is used in order to obtain objective stress rate measurements.

2 THE REDUCED INTEGRATION AND THE STABILIZING PROCEDURE

The equation of motion in the FEM context may be expressed at the element level as follows:

\[ \mathbf{M} \ddot{\mathbf{U}} + \mathbf{D} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} = \mathbf{P} \]

with:

\[ \mathbf{M} = \int_{V_e} \rho \mathbf{N}^t \mathbf{N} \, dV ; \quad \mathbf{D} = \int_{V_e} \varphi \mathbf{N}^t \mathbf{N} \, dV ; \quad \mathbf{K} = \int_{V_e} \mathbf{B}^t \mathbf{C} \mathbf{B} \, dV ; \quad \mathbf{P} = \int_{V_e} \mathbf{N}^t \mathbf{b} \, dV + \int_{S_e} \mathbf{N}^t \mathbf{p} \, dS \]

where \( \mathbf{M} \), \( \mathbf{D} \) and \( \mathbf{K} \) are the mass, damping and stiffness matrices, respectively, and \( \mathbf{P} \) is the load vector, which is constituted by body (\( \mathbf{b} \)) and surfaces (\( \mathbf{p} \)) forces acting on volume \( V_e \) and surface \( S_e \), respectively. \( \mathbf{U} \), \( \dot{\mathbf{U}} \) and \( \ddot{\mathbf{U}} \) are the displacement, velocity and acceleration vectors, respectively, and \( \mathbf{C} \) is the elastic constitutive matrix. In addition, \( \mathbf{N} \) is the vector containing the shape functions and \( \mathbf{B} \) is the gradient matrix relating strain and displacement components. Finally, \( \rho \) and \( \varphi \) are the specific mass and the damping coefficient, respectively.

Consider a tri-linear isoparametric hexahedral element where uniform one-point quadrature is used so that the shape functions and its derivatives are evaluated at the center of the element. In order to avoid volumetric locking the uniform reduced integration must be performed over the gradient matrix \( \mathbf{\bar{B}} \). This matrix is then decomposed as follows:

\[ \mathbf{\bar{B}}(\xi, \eta, \zeta) = \mathbf{\bar{B}}(0) + \mathbf{\bar{B}}(\xi, \eta, \zeta) \]

where \( \mathbf{\bar{B}}(0) \) is the part of the gradient matrix that corresponds to the volumetric part of the strain vector evaluated at the center of the element and \( \mathbf{\bar{B}}(\xi, \eta, \zeta) \) is the part of the gradient matrix that corresponds to the deviatoric part of the strain vector, where \( \xi \), \( \eta \) and \( \zeta \) are the
natural coordinates.

Expanding $\hat{B}(\xi, \eta, \zeta)$ and the stress vector $\sigma(\xi, \eta, \zeta)$ in Taylor series at the element center up to bilinear terms and substituting these expansions into the expression of the virtual work, considering the decomposition given by Eq. (3), the following expression for the internal virtual work at element level is obtained:

$$\delta W_{int} = \delta U_{(e)}^T \left[ B'(0)\sigma(0) + \frac{1}{3} \hat{B}'_{,\xi}(0)\tilde{\sigma}_{,\xi}(0) + \frac{1}{3} \hat{B}'_{,\eta}(0)\tilde{\sigma}_{,\eta}(0) + \frac{1}{3} \hat{B}'_{,\zeta}(0)\tilde{\sigma}_{,\zeta}(0) + \frac{1}{9} \hat{B}'_{,\xi\eta}(0)\tilde{\sigma}_{,\xi\eta}(0) + \frac{1}{9} \hat{B}'_{,\xi\zeta}(0)\tilde{\sigma}_{,\xi\zeta}(0) + \frac{1}{9} \hat{B}'_{,\eta\zeta}(0)\tilde{\sigma}_{,\eta\zeta}(0) \right] V_{(e)}$$

(4)

where $V_{(e)}$ is the volume of the element $(e)$ and $B(0) = \hat{B}(0) + \hat{B}(0)$ is the contribution of the volumetric and deviatoric parts of the gradient matrix obtained from one-point integration. The same notation is valid for stresses.

The internal force vector at element level may be expressed by:

$$f_{int} = f^e + f^{hg} = (K^e + K^{stab})U$$

(5)

where:

$$f^e = B'(0)\sigma(0)V_e = \left[ B'(0)CB(0)V_{e} \right] U = K^e U$$

(6)

which corresponds to the one-point quadrature part of the internal force vector and:

$$f^{hg} = K^{stab}U = \left[ \frac{1}{3} \hat{B}'_{,\xi}(0)\tilde{\sigma}_{,\xi}(0) + \frac{1}{3} \hat{B}'_{,\eta}(0)\tilde{\sigma}_{,\eta}(0) + \frac{1}{3} \hat{B}'_{,\zeta}(0)\tilde{\sigma}_{,\zeta}(0) + \frac{1}{9} \hat{B}'_{,\xi\eta}(0)\tilde{\sigma}_{,\xi\eta}(0) + \frac{1}{9} \hat{B}'_{,\xi\zeta}(0)\tilde{\sigma}_{,\xi\zeta}(0) + \frac{1}{9} \hat{B}'_{,\eta\zeta}(0)\tilde{\sigma}_{,\eta\zeta}(0) \right] V_{(e)}$$

(7)

which corresponds to the hourglass control part of the internal force vector. The stress-strain constitutive relations for the first and second derivatives of the stress vector in Eq. (7) are obtained as follows:

$$\tilde{\sigma}_{,\xi} = E\tilde{\varepsilon}_{,\xi}; \tilde{\sigma}_{,\eta} = E\tilde{\varepsilon}_{,\eta}; \tilde{\sigma}_{,\zeta} = E\tilde{\varepsilon}_{,\zeta}; \tilde{\sigma}_{,\xi\eta} = E\tilde{\varepsilon}_{,\xi\eta}; \tilde{\sigma}_{,\xi\zeta} = E\tilde{\varepsilon}_{,\xi\zeta}; \tilde{\sigma}_{,\eta\zeta} = E\tilde{\varepsilon}_{,\eta\zeta}$$

(8)

where $E$ is the stabilization matrix, proposed by Hu and Nagy (1997), which is given by:

$$E = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} = 2\mu I$$

(9)

with $\mu$ being the shear modulus and $I$ being the identity matrix.

In order to remove shear locking, shear components of the strain tensor must be written in an orthogonal co-rotational coordinates system. In addition, all shear components must be linearly interpolated in a single coordinate direction of the reference system. Then:

$$\varepsilon_{,\xi}(\xi, \eta, \zeta) = \varepsilon_{,\xi}(0) + \hat{\varepsilon}_{,\xi}(0)\zeta$$

$$\varepsilon_{,\eta}(\xi, \eta, \zeta) = \varepsilon_{,\eta}(0) + \hat{\varepsilon}_{,\eta}(0)\xi$$

$$\varepsilon_{,\zeta}(\xi, \eta, \zeta) = \varepsilon_{,\zeta}(0) + \hat{\varepsilon}_{,\zeta}(0)\eta$$

(10)
Consequently:
\[
\hat{\mathbf{B}}_{\alpha;\xi}(0) = \hat{\mathbf{B}}_{\alpha;\eta}(0) = \hat{\mathbf{B}}_{\alpha;\phi}(0) = \hat{\mathbf{B}}_{\alpha;\psi}(0) = \hat{\mathbf{B}}_{\alpha;\chi}(0) = 0
\]
\[
\hat{\mathbf{B}}_{\beta;\eta}(0) = \hat{\mathbf{B}}_{\beta;\phi}(0) = \hat{\mathbf{B}}_{\beta;\psi}(0) = \hat{\mathbf{B}}_{\beta;\chi}(0) = 0
\]
\[
\hat{\mathbf{B}}_{\gamma;\phi}(0) = \hat{\mathbf{B}}_{\gamma;\psi}(0) = \hat{\mathbf{B}}_{\gamma;\chi}(0) = \hat{\mathbf{B}}_{\gamma;\eta}(0) = 0
\]  

(11)

The internal force vector will not be adequately evaluated for distorted elements if one-point quadrature is used. In order to correct this deficiency the gradient matrix obtained with reduced integration \( \mathbf{B}(0) \) must be replaced by uniform gradient submatrices \( \mathbf{B}_a'(0) \), defined by Flanagan and Belytschko (1981) as follows:

\[
\mathbf{B}_a'(0) = \frac{1}{V_e} \int_{V_e} \mathbf{B}_a(\xi, \eta, \zeta) \, dV \quad (a = 1, \ldots, 8)
\]

(12)

where \( a \) corresponds to the node number of the element \( e \).

3 THE GEOMETRICALLY NONLINEAR ANALYSIS WITH THE CO-ROTATIONAL REFERENCE SYSTEM

In order to avoid shear locking, it is also necessary to employ a co-rotational coordinates system for the geometric description, where the reference system is attached to the local coordinates system of the finite element.

The motion of a continuous medium may be decomposed into rigid body motion and pure deformation. Since the spatial discretization of the problem is fine enough, this decomposition can be performed at element level, and consequently, in the co-rotational system, where the pure deformation portion will always be a small quantity relative to the element dimensions.

To calculate stress and strain updates, it is necessary to evaluate the deformation part of the displacement field in the co-rotational system. The displacement field, in an incremental form, can be separated into a part owing to pure deformations and a part owing to pure rotations, similarly to polar decomposition. The incremental displacement due to pure rotation may be obtained from:

\[
\Delta \mathbf{u}^{\text{rot}} = \Delta \mathbf{u} - \mathbf{R}^t_{n+1/2} \left( \mathbf{\hat{x}}_{n+1} - \mathbf{\hat{x}}_n \right)
\]

(13)

where \( \mathbf{R}^t_{n+1/2} \) is the transpose of the orthogonal transformation matrix \( \mathbf{R}_{n+1/2} \), which rotates the global coordinate system to the corresponding co-rotational coordinates system. This matrix is referred to the mid-point of the time interval \([t_n, t_{n+1}]\) and it is defined from the geometric configuration at this point. \( \mathbf{\hat{x}}_n \) and \( \mathbf{\hat{x}}_{n+1} \) are, respectively, the geometric configurations at \( t = t_n \) and \( t = t_{n+1} \) defined in the co-rotational coordinates system. The deformation displacement increment in the co-rotational system is finally obtained by the expression:

\[
\Delta \mathbf{u}^{\text{def}} = \mathbf{R}^t_{n+1/2} \Delta \mathbf{u}^{\text{def}} = \mathbf{\hat{x}}_{n+1} - \mathbf{\hat{x}}_n
\]

(14)

The components of the transformation matrix are given by:

\[
\mathbf{R}_{ij} = \frac{r_{ij}}{r_i r_j} \quad ; \quad \mathbf{R}_{3j} = \frac{(r_{2j} + r_{3j})}{(r_{2j} + r_{3j})^j(r_{2j} + r_{3j})} \quad ; \quad \mathbf{R}_{3j} = \frac{r_{3j}}{r_i r_j} \quad (j = 1, 2, 3)
\]

(15)

with:
\[ \mathbf{r}_{ij} = \xi^i \mathbf{x}_j \; ; \; \mathbf{r}_{2j} = \eta^i \mathbf{x}_j \; ; \; \mathbf{r}_{ij} = -\frac{\mathbf{r}_{ij}^r \mathbf{r}_{2j}}{r_{ij}^r} \; ; \; \mathbf{r}_{3j} = \mathbf{r}_{ij} \times (\mathbf{r}_{2j} + \mathbf{r}_{ij}) \]  

(16)

where \( \mathbf{x}_j \) is the vector of nodal global coordinates of a generic element and \( \xi^i \) and \( \eta^i \) are vectors containing natural coordinates.

The strain increment is given by:

\[ \Delta \mathbf{\varepsilon} = \frac{1}{2} \left[ \frac{\partial \Delta \mathbf{u}^{\text{def}}}{\partial \xi^{n+1/2}} + \left( \frac{\partial \Delta \mathbf{u}^{\text{def}}}{\partial \xi^{n+1/2}} \right)^T \right] \]  

(17)

The co-rotational Cauchy stress tensor is used as stress measure since it is an objective tensor in this coordinate system. However, the rate of the Cauchy tensor is not objective. Hence, the Truesdell stress rate is employed in this work, which is given in its incremental form by:

\[ \Delta \sigma_{ij} = \left( C_{ijkl} + \hat{C}_{ijkl} \right) \Delta \mathbf{e}_{kl} + W_{ijkl} \Delta \omega_{kl} \quad (i,j,k,l=1,2,3) \]  

(18)

where:

\[ \hat{C}_{ijkl} = -\sigma_{ij} \delta_{kl} + \frac{1}{2} \left( \sigma_{il} \delta_{jk} + \sigma_{jl} \delta_{ik} + \sigma_{ik} \delta_{jl} + \sigma_{jk} \delta_{il} \right) \]  

(19)

and:

\[ W_{ijkl} = \frac{1}{2} \left( \sigma_{ij} \delta_{jk} - \sigma_{jl} \delta_{ik} - \sigma_{ik} \delta_{jl} + \sigma_{jk} \delta_{il} \right) \]  

(20)

with \( \delta_{ij} \) being the Kroenecker delta.

4 TIME DISCRETIZATION PROCEDURES

4.1 The Generalized-\( \alpha \) method with optimized time integration parameters

The Newmark scheme is an extension of the linear acceleration method in which a linear variation of acceleration is assumed in the time interval \([t, t+\Delta t]\). Further information about the Newmark method may be found, for instance, in Bathe (1996). Although this scheme is unconditionally stable for linear problems, it may be unstable for nonlinear problems.

The Generalized-\( \alpha \) method includes the most popular numerical algorithms employed in the time discretization of the dynamic equation (Newmark method, presented by Newmark, 1959; Hilber-\( \alpha \) method, presented by Hilber et al., 1977; Bossak-\( \alpha \) method, presented by Wood et al., 1981). The method was originally developed by Chung and Hulbert (1993) for linear structural dynamics combining second-order accuracy, minimal numerical dissipation of lower modes and maximal numerical dissipation of higher modes. It was extended to nonlinear problems by Kuhl and Ramm (1996), determining optimal algorithmic parameters for the time integration process.

In the Generalized-\( \alpha \) method the equilibrium equation of motion is verified at a general point within the time interval instead of the end-point \( n+1 \) used by the classical Newmark scheme. Consequently, the modified equation of motion may be written in the following form:
\[
M \dddot{U}_{n+1} + D \ddot{U}_{n+1} \alpha_f + f_{n+1}^{int} = P_{n+1} \alpha_f
\]  

where:

\[
f_{n+1}^{int} = (1 - \alpha_f) f_{n+1}^{int} + \alpha f_{n}^{int} = (1 - \alpha_f) f_{n+1}^{int}(U_{n+1}) + \alpha f_{n}^{int}(U_n)
\]

\[
\ddot{U}_{n+1} = (1 - \alpha_m) \ddot{U}_{n+1} + \alpha_m \ddot{U}_n
\]

\[
\dot{U}_{n+1} = (1 - \alpha_f) \dot{U}_{n+1} + \alpha f \dot{U}_n
\]

\[
U_{n+1} = (1 - \alpha_f) U_{n+1} + \alpha f U_n
\]

\[
P_{n+1} = (1 - \alpha_f) P_{n+1} + \alpha f P_n
\]

In expressions (21) and (22) subscripts denote positions where variables are evaluated within the time interval \([t, t+\Delta t]\). These positions are defined by the time integration parameters \(\alpha_m\) and \(\alpha_f\), while \(n\) and \(n+1\) indicate initial and end points of the time interval, respectively.

Introducing Newmark approximations for \(U_{n+1}\), \(\dot{U}_{n+1}\) and \(\ddot{U}_{n+1}\) in Eqs. (22), and then employing these expressions into the modified equation of motion (Eq. 21), the effective dynamic equation can be obtained, which has the following form:

\[
\left\{ \left[ \frac{1 - \alpha_m}{\alpha m} \right] M + \left[ \frac{(1 - \alpha_f) \delta}{\alpha m} \right] D + \left[ \frac{1 - \alpha_m - 2 \alpha_m}{2 \alpha} \right] K_{n+1}^{\text{int}} \right\} \Delta U =
\]

\[
P_{n+1} = f_{n+1}^{int} (U_{n+1}) - M \left[ \frac{1 - \alpha_m}{\alpha m} \ddot{U}_n + \frac{1 - \alpha_m - 2 \alpha_m}{2 \alpha} \dddot{U}_n \right] -
D \left[ \frac{(1 - \alpha_f) \delta - \alpha}{\alpha} \dot{U}_n + \frac{(\delta - 2 \alpha)(1 - \alpha_f)}{2 \alpha} \Delta t \ddot{U}_n \right]
\]  

where:

\[
K_{n+1}^{\text{int}} = (1 - \alpha_f) K^{\text{int}}(U_{n+1}^k)
\]

In the Generalized-\(\alpha\) method the time integration parameters \(\alpha\), \(\delta\), \(\alpha_m\) and \(\alpha_f\) are defined as functions of the spectral radius \(r_\alpha\) (0 \(\leq r_\alpha \leq 1\)) as follows (see Kuhl and Crisfield, 1999):

\[
\alpha = \frac{1}{4} \left( (1 - \alpha_m + \alpha_f) \right)^2 ; \quad \delta = \frac{1}{2} \alpha_m + \alpha_f ; \quad \alpha_m = \frac{2 r_\alpha - 1}{r_\alpha + 1} ; \quad \alpha_f = \frac{r_\alpha}{r_\alpha + 1}
\]

The numerical procedure to obtain displacements components by the Generalized-\(\alpha\) method given in Eq. (23) is similar to the Newmark algorithm for nonlinear analyses (see Bathe, 1996).

4.2 The algorithmic energy control scheme

In this work a modification in the Generalized-\(\alpha\) method is proposed in order to obtain a numerical scheme with controllable numerical dissipation and energy-conserving time integration. A sufficient condition to keep a stable time integration process in the nonlinear regime is the conservation or decay of the total energy within a time step. This energy criterion may be represented by the following inequality (see Belytschko and Schoeberle, 1975):
\[ \Delta U + \Delta K - W_{\text{ext}} \leq 0 \] (26)

where \( W_{\text{ext}} \) is the work done by external forces within the time step and \( \Delta U = U_{n+1} - U_n \) and \( \Delta K = K_{n+1} - K_n \), in which \( U_{n+1} \) and \( U_n \) are the strain energies at the beginning and the end of the time step and \( K_{n+1} \) and \( K_n \) are the corresponding kinetic energies. It is assumed that heat is not created or dissipated by the mechanical work.

If the energy criterion is to be perfectly satisfied, the work done by external forces must be exactly equilibrated by the sum of strain and kinetic energy increments. Otherwise, a residual energy \( R \) will exist. This residual energy is given by:

\[ R = W_{\text{ext}} - (\Delta U + \Delta K) \] (27)

where \( R \), which may be a lack or excess of the structural system energy, must be minimized. In the present work, it is suggested that the energy to be added or removed from the structural system, in order to satisfy the energy restriction (Eq. 26) exactly, may be approximated by the mechanical work done by a fictitious viscous force term as it is shown below:

\[ \beta = \frac{R}{U_{n+1}^\top M U_{n+1}} \] (28)

where \( D_f \) is an artificial damping matrix which is added to the damping term of the equation of motion (Eq. 1) and it is continually modified along the Newton-Raphson iterative process according to energy balance conditions of the corresponding time step.

### 4.3 The tangent stiffness matrix and the internal force vector in the nonlinear analysis

In the geometrical nonlinear regime the dynamic equilibrium of the structure must be iteratively satisfied using the incremental approach, where the stiffness matrix and the internal force vector are considered as functions of the current body configuration. The discretized form of the equilibrium equation, taking into account the energy control scheme presented above, may be described according to the following expression (Mondkar and Powell, 1977):

\[ M \Delta \dot{U} + \left( D + D' \right) \Delta \dot{U} + K_{t}^{\text{tang}} \Delta U = P_{t+\Delta t} - \left[ f_{\text{int}} \left( U \right) + M \ddot{U} + \left( D + D' \right) \dot{U} \right] \] (29)

where \( \Delta \ddot{U} \), \( \Delta \dot{U} \) and \( \Delta U \) are vectors containing incremental values of acceleration, velocity and displacement components, respectively. Equation (29) was obtained using a linearization procedure given by the Newton-Raphson method (see Bathe, 1996).

The dynamic equilibrium must be reached using an iterative process. Consequently, the displacement field at the current time is obtained from the incremental solution \( \Delta U \). The incremental solution \( \Delta U \) produces a new approximation for \( U_{t+\Delta t} \), generating a new state of stresses for the structural system, which must be in equilibrium with external forces. Therefore, the iterative process in each time interval goes on until the equilibrium is obtained, considering a given tolerance criterion.

The tangent stiffness matrix and the internal force vector are evaluated in the co-rotational coordinates system at the time instant \( t \) and iteration \( j \) by:

\[ K_{t}^{\text{tang,c}} = \int_{\Omega} B^\top (C + \dot{C}_j) B d\Omega \] ; \[ f_{t}^{\text{int,c}} = \int_{\Omega} B^\top \sigma_j d\Omega \] (30)
where $\Omega^c_j$ is referred to the body configuration in the co-rotational system at the iteration $j$ and $\hat{C}_j$ and $\sigma_j$ are stress tensors related to the Truesdell rate and the co-rotational Cauchy stress tensor, respectively (see Belytschko et al., 2000).

The tangent stiffness matrix and the internal force vector in the global system are obtained, respectively, according to the following objective transformation from the co-rotational system:

$$
K^{\text{tang}} = R^T K^{\text{tang},c} R \quad ; \quad f^{\text{int}} = R^T f^{\text{int},c}
$$

(31)

where $R$ is the orthogonal transformation matrix of the co-rotational system, which is locally evaluated with equations (15) and (16) at element level.

5 NUMERICAL APPLICATIONS

In this section three representative simulations are presented in order to validate the proposed algorithm for geometrically nonlinear dynamic analysis and also to demonstrate the accuracy and applicability of the element formulation. All simulations are carried out considering geometrically nonlinear effects and the time integration is performed using the classical Newmark scheme and the Generalized-$\alpha$ method with algorithmic energy control. The following definitions are valid for all examples analyzed in this work: $E_{\text{pot}}$ – strain energy, $E_{\text{kin}}$ – kinetic energy, $E_{\text{tot}}$ – total energy, $W_{\text{ext}}$ – work done by external forces, $J_x$ – angular momentum around $x$ axis, $J_y$ – angular momentum around $y$ axis and $J_z$ – angular momentum around $z$ axis. These variables may be obtained with the following expressions:

$$
J = \{J_x, J_y, J_z\} = \sum_{V_e} \int_{V_e} \rho \mathbf{x} \otimes \dot{\mathbf{x}} \, dV_e
$$

(32)

$$
E_{\text{kin}} = \sum_{V_e} \frac{1}{2} \int_{V_e} \rho \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \, dV_e
$$

(33)

$$
E_{\text{pot}} = \sum_{V_e} \frac{1}{2} \int_{V_e} \mathbf{\sigma} : \mathbf{\varepsilon} \, dV_e
$$

(34)

where the symbols ($\otimes$), ($\cdot$) and ($\otimes$) denote tensorial, scalar and vectorial products, respectively. The vectors $\mathbf{x}$ and $\dot{\mathbf{x}}$ are the position vector in global coordinates and its respective time derivative, $\text{nel}$ is the total number of elements, $V_e$ is the element volume and $\mathbf{\sigma}$ and $\mathbf{\varepsilon}$ are stress and strain energetically conjugated tensors. It is worth to notice that Eq. (34) is valid for Saint Venant-Kirchhoff materials only.

5.1 Cantilever beam

A 2D cantilever beam subjected to pressure loading and undergoing large displacements is analyzed. Geometrical and load description for the present simulation are shown in Fig. 1 and material properties of the structure as well as the time step employed in the time integration are found in Table 1. The computational domain is modeled with 800x4x1 eight-node hexahedral elements considering a plane strain state. The following values are taken for the spectral radius: $r_x = 0.4$, $r_y = 0.3$, $r_z = 0.2$, $r_x = 0.1$ and $r_y = 0.0$.

The structural responses of the beam at its tip using the Newmark’s method and the new formulation presented in this work are shown in Fig. 2, Fig. 3 and Fig. 4. The Newmark’s method was unable to reproduce the dynamic response of the beam owing to the lack of
conservation properties of the numerical algorithm, as it was expected. The numerical instability is clearly observed in the acceleration time history, where a growing amplitude response is obtained, leading to the breakdown of the time integration process. Referring to the Generalized-$\alpha$ method with energy control, it is observed that unstable time integration is also verified in the range $0.4 \leq r_\infty \leq 1.0$ due to the highly nonlinear behavior of the structure. Stable responses are obtained in the range $0.0 \leq r_\infty < 0.4$, where dissipation properties of the numerical scheme can damp out spurious oscillations of the numerical results. The structural response obtained in the present work is very similar to that presented by Bathe and Baig (2005).

![Figure 1. Geometrical and load characteristics for the cantilever beam analysis.](image)

<table>
<thead>
<tr>
<th>Young modulus – $E$ [N/m$^2$]</th>
<th>$7 \times 10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson coefficient – $\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>Specific mass – $\rho$ [Kg/m$^3$]</td>
<td>$2.7 \times 10^3$</td>
</tr>
<tr>
<td>Damping coefficient – $\varphi$</td>
<td>0.0</td>
</tr>
<tr>
<td>Time step – $\Delta t$ [s]</td>
<td>$5 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 1. Material properties and time step for the cantilever beam analysis.

Time histories of the energy variables, which are present in the energy constraint equation (Eq. 26), are plotted in Fig. 5. As it was stated earlier, the Newmark’s method loses the unconditional stable characteristic in the nonlinear regime owing to the lack of conservation or decrease of the total energy. This conclusion is verified in the unstable results presented below, where the total energy blows up suddenly. On the other hand, stable solutions are obtained by the Generalized-$\alpha$ method with energy control with minimal numerical damping. When compared to other dissipation-based algorithms, the scheme presented in this work is not disturbed by excessive numerical damping, which leads to excessive decrease of the energy and motion variables (displacement, velocity and acceleration).
Figure 2. Displacement response for the cantilever beam analysis using the Newmark’s method and the Generalized-α method with algorithmic energy control.
Figure 3. Velocity response for the cantilever beam analysis using the Newmark’s method and the Generalized-\(\alpha\) method with algorithmic energy control.
Figure 4. Acceleration response for the cantilever beam analysis using the Newmark’s method and the Generalized-\(\alpha\) method with algorithmic energy control.

Figure 5. Energy response for the cantilever beam analysis using the Newmark’s method and the Generalized-\(\alpha\) method with algorithmic energy control.
5.2 Toss rule in plane

The geometrically nonlinear dynamic analysis of a toss rule is performed in this example. Geometry and load information for the simulations carried out here are described in Fig. 6 and material properties of the structure as well as the time step used in the time integration are presented in Table 2. It is important to notice that distributed loads are applied to the structure to produce the plane motion of the rule, which is free to fly in the absence of displacement restrictions and gravity action. Simulations employing the Generalized-α method with energy control are executed using the following spectral radii: $r_\alpha = 1.0$, $r_\alpha = 0.9$, $r_\alpha = 0.8$, $r_\alpha = 0.6$, $r_\alpha = 0.4$, $r_\alpha = 0.2$ and $r_\alpha = 0.0$. The finite element mesh is constituted by 480 (30x4x4) eight-node hexahedral elements with one-point integration.

![Diagram](image)

**Figure 6. Geometrical and load characteristics for the motion analysis of the toss rule in plane.**

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young modulus – $E$ [N/m²]</td>
<td>$2.06 \times 10^{11}$</td>
</tr>
<tr>
<td>Poisson coefficient – $\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Specific mass – $\rho$ [Kg/m³]</td>
<td>$7.8 \times 10^3$</td>
</tr>
<tr>
<td>Damping coefficient – $\varphi$</td>
<td>0.0</td>
</tr>
<tr>
<td>Time step – $\Delta t$ [s]</td>
<td>$5 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

**Table 2. Material properties and time step for the motion analysis of the toss rule in plane.**

In Fig. 7 the evolution of the different energy variables considered in the vibration process over the time integration period is displayed. The Newmark’s method was unable to simulate the motion of the rule by the same reasons mentioned before, showing the characteristic increase of the total energy response and the collapse of the numerical integration during the iteration process. The Generalized-α method with energy control and $r_\alpha = 1.0$ leads also to numerical failure, since the value used for the spectral radius corresponds to the case of no algorithmic damping. However, a stable solution can be obtained even with small algorithmic dissipation ($r_\alpha = 0.9$). A small decrease in the total energy is observed only for $r_\alpha = 0.0$, when maximum dissipation is prescribed to the numerical scheme. On the other hand, it is clearly observed that a small amount of numerical dissipation (larger $r_\alpha$) is sufficient to stabilize the spurious oscillations. Another important advantage of the present formulation is referred to
the exact conservation of angular momentum, which is also necessary to indicate algorithmic stability of the numerical scheme. The range of energy values obtained with the present algorithm is in agreement with those obtained by Khul and Ramm (1996).

Some deformed configurations obtained with the algorithm proposed in this paper are shown in Fig. 8, where the inertial motion is observed after the initial load is removed and structural displacements take place on the plane $x$-$z$ owing to the initial load configuration.
Figure 7. Energy response for motion analysis of the toss rule in plane.

\[ \Delta T = 0.001 \text{ s} \]

\[ t = 0.001 \text{ s} \quad t = 0.03 \text{ s} \]

\[ \Delta T = 0.01 \text{ s} \]

\[ t = 0.01 \text{ s} \quad t = 0.1 \text{ s} \]

Figure 8. Deformed configurations of the rule during time integration. Up: plane view; Down: spatial view
5.3 Toss rule in space

\[ h = 0.002 \, \text{m}; \, l = 0.3 \, \text{m}; \, b = 0.06 \, \text{m} \]

Figure 9. Geometrical and load characteristics for the motion analysis of the toss rule in space.

In this section the toss rule analysis performed in the previous example is slightly modified in order to demonstrate the performance of the algorithm proposed in this paper for three-dimensional motions. Geometry and load information are shown in Fig. 9, where additional loads are applied to the structure to enable the movement in space. Material properties and time step of the structure are the same as in the preceding analysis (see Table 2). The rule has no restrictions to the displacement field and it is free to fly in the absence of gravity. Simulations employing the Generalized-\( \alpha \) method with energy control are executed using the following spectral radii: \( r_x = 1.0, \, r_x = 0.9, \, r_x = 0.8, \, r_x = 0.6, \, r_x = 0.4, \, r_x = 0.2 \) and \( r_x = 0.0 \). The same finite element mesh of the previous section is used here.

In Fig. 10 time histories of energy variables are shown over the time integration period considered for the present example. The Newmark’s method cannot avoid numerical instabilities due to numerical dissipation deficiencies in the algorithm, as in previous analyses. The Generalized-\( \alpha \) method with energy control and \( r_x = 1.0 \) leads to the same catastrophic behavior presented by the Newmark’s scheme, which is related to the lack of algorithmic dissipation. When the numerical dissipation is large enough (\( r_x \leq 0.9 \)), the proposed algorithm can obtain stable solutions, whereas Generalized-\( \alpha \) schemes require larger numerical damping to stabilize the numerical integration. Exact conservation of angular momentum is demonstrated again in the present simulation. The energy responses obtained in this section with the presented algorithm are in agreement with those obtained by Khul and Ramm (1999).
Figure 10. Energy response for the motion analysis of the toss rule in space.

Deformed shapes of the rule obtained with the algorithm proposed in this paper are shown in Fig. 11 on snap shots. Resultant forces of the load set lead to complex inertial motion with
translational and rotational displacements along to torsional, shear and bending deformations in three directions.

Figure 11. Deformed configurations of the rule during time integration, spatial view.

6 CONCLUSIONS

The main objective of the present work was the validation of a modification in the Generalized-α scheme formulation for geometrically nonlinear dynamic analysis of elastic structures. It was observed that the proposed time discretization algorithm worked finely as well as the element formulation. As it is expected, the Newmark’s method did not guarantee stable time integration in nonlinear elastodynamics, especially for highly nonlinear problems. On the other hand, the Generalized-α method with energy control led to stable solutions when appropriate numerical dissipation was used. Conservation of energy and angular momentum were shown through the examples. The algorithm keeps also the great advantage of the dissipation-based models, which is referred to the low computational cost of one-step schemes (similar to the classical Newmark’s scheme). It was also observed that smaller spectral radii reduce the number of average iteration and, consequently, the running time per time step, whereas spurious oscillations are more damped in this case owing to higher levels of numerical dissipation, especially for complex motions. Otherwise, these values may be augmented if excessive dissipation is provided to the scheme, as it is verified in the motion analysis of the toss rule in plane. It was shown the good performance of the 3-D hexahedral element for highly nonlinear dynamic problems involving thin elastic plates. Element matrices may be obtained analytically when one-point quadrature is used and the number of degrees of freedom per node is reduced with respect to classical elements used for shells and plates. A drawback of the present model is that shell and plate analyses require usually a greater number of elements in the spatial discretization than classical formulations because the one-point quadrature technique is employed. An important implementation for the numerical model presented here, which is concerned to a more extended applicability of the present formulation, refers to the treatment of general material models in Elasticity. Therefore, the scheme shown in this paper must be re-formulated in future works in order to take into account Hyperelastic material models (for instance, a Neo-Hookean model).

REFERENCES


