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Oscillations in a nonlinear dispersive system of equations.

Numerical experiments are presented that illustrate the emergence of oscillatory behavior in the solutions of a semi-discrete dispersive scheme approximating the Hopf equation. The oscillations arise at the same time that the classical solution of the Hopf equation develops a singularity and generally have a spatial period equal to twice the grid size.

1. Introduction

We examine the finite amplitude oscillations arising in a dispersive semi-discrete "numerical" approximation to the Hopf equation initial value problem (1): $u_t + (u^2)_x = 0$, $u(x,0) = f(x)$, where $f = f(x)$ is (generally) smooth. The "approximation" to be considered is given by the following semi-discrete (continuous in time, discrete in space) dispersive difference scheme (2): $\dot{u}_n + \frac{1}{2h}\left(u_{n+1}^2 - u_{n-1}^2\right) = 0$, where n is an integer index, $0 < h \ll 1$, $u_n(0) = f(x_n)$, $x_n = nh + x_0$ and x_0 is some arbitrary fixed constant. Expanding $u_{n\pm1}$ in a Taylor series centered at x_n, the leading order truncation error for the scheme is found to be nonlinear dispersive $u_t + \left(u^2\right)_x + h^2\left(\frac{1}{6}u^2\right)_{xxx} = O\left(h^4\right)$. It follows that the continuum limit, $h \to 0$ should present many similarities with the well understood zero dispersion limit, $\epsilon \to 0$, for the Korteweg-de Vries initial value problem $u_t + (u^2)_x + \epsilon^2 u_{xxx} = 0$, $u(x,0) = f(x)$. The behavior of the solution $u = u(x,t;\epsilon)$ in the limit $\epsilon \to 0$ is well known ([1], [2] and [3]): As long as the solution of equation (1) is smooth, $u(x,t;\epsilon)$ tends uniformly to that smooth solution. However, when t exceeds the time after which the solution of (1) breaks down, $u(x,t;\epsilon)$ develops oscillations over some x-interval near the breakdown point. As ϵ vanishes the amplitude of the oscillations remains finite and their wavelength is $O(\epsilon)$. The nature of the oscillations can be predicted rather precisely in this case.

2. The modulation equations for binary oscillations

Here we derive the Modulation Equations for binary oscillations and study the equations. An exact analytical solution is provided for the case of step initial data. Consider the limit $h \to 0$ of (2), when the odd and even u_n are approximated by two distinct smooth functions $a(x_n,t)$ and $b(x_n,t)$ — modulated binary oscillations. Expanding in Taylor series around x_n any function evaluated at $x_{n\pm1}$ and collecting powers of h we obtain, at leading order, the equations (3): $a_t + (b^2)_x = 0$ and $b_t + (a^2)_x = 0$, with initial conditions $a(x,0) = b(x,0) = f(x)$. Given the fact that a and b are equal initially, one may wonder how it ever develops that $a \neq b$. This occurs because "shocks" are allowed in (3). A steady discontinuity across which the binary oscillations simply change their sign is consistent with the continuum limit of (2). Thus (3) allows for steady shocks, across which a and b change signs. After a smooth solution to (1) breaks down — and if the solution is odd around the point of breakdown — binary oscillations, governed by (3), will arise in (2) (with a steady shock at the point of breakdown and an expansion fan on each side.) The mechanism is made clear by the exact solution (4) below. The system (3) is hyperbolic when $ab > 0$, has characteristic speeds $\lambda_\pm = \pm 2\sqrt{ab}$ and can be written in Riemann Invariant form. When $ab < 0$ the system is elliptic. Consider now the problem for (3) on $t > 0$ and $-\infty < x < \infty$, with initial data $a(x,0) = b(x,0) = -\text{sign}(x)$. The solution will be odd, with a steady shock at $x = 0$. But just stating that the jumps keep the squares continuous is not enough to determine the solution uniquely. An additional condition at the shock (determined by the details of the continuum limit in (2), that we study elsewhere —see [4]) is needed. Here we take the simplified form $b(0,t) = \alpha a(0,t)$, for some given constant $0 \leq \alpha \leq 1$. In general the constant α depends on x_0 above in (2). Thus the occurrence of binary oscillations is strongly phase (placement of the grid relative to the initial conditions function $f(x)$, given by x_0) dependent: for $\alpha = 1$ no oscillations occur and for $\alpha = 0$ maximum amplitude oscillations are generated. The solution, indicating the generation of oscillations by the discontinuity, is then given by (4): $a(x,t) = b(x,t) = 1$ for $x \leq -2t$, $\quad a = (1+S)^{2/3}$ and $b = (1-S)^{2/3}$ for $-2t \leq x \leq -2\beta t$, $a = \beta/\sqrt{\alpha}$ and $b = \alpha a$ for $-2\beta t \leq x \leq 0$, $\quad$ where $S = \sqrt{1 + (x/2t)^3}$ and $\beta = [4\alpha^{1.5}/(1+\alpha^{1.5})^2)]^{1/3}$.

3. Some numerical experiments

In this section we will consider an example of initial data for the problem (2) and integrate the equations numerically. The data are such that the solution is odd around the breakdown point for the smooth solution of (1). The aim is to

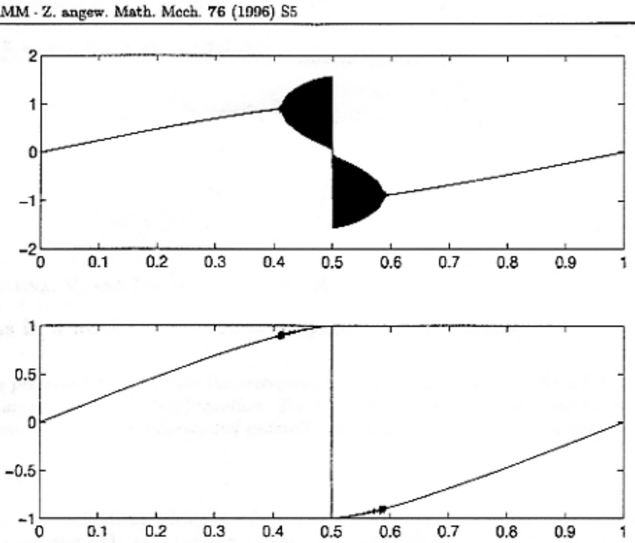

Figure 1: $u_n(0) = sin(2\pi x_n)$ for $x_0 = 0$ and $h = 1/1000$ (upper) or $h = 1/1001$ (lower) at time $t = 0.13$.

illustrate the general mechanism introduced in Section 2 for the formation of steady shocks and binary oscillations. For the initial data we take $u_n(0) = \sin(2\pi x_n)$, $x_n = nh$, where $h = 1/N$ and $x_0 = 0$. The equations are then integrated using periodic boundary conditions. We show here the results of two calculations. One in the upper figure with $N = 1000$ and the other in the lower figure with $N = 1001$. A simple symmetry argument shows that $N = 1000$ corresponds to $\alpha = 0$ at the shock, while $N = 1001$ gives $\alpha = 1$. The numerical results are strikingly different and confirm this. For $N = 1000$ we see period two oscillations arising at $x = 0.5$ at a time $t \approx t_B = 1/4\pi$ and fanning out in both directions — to fill a region of width $\Delta x = O(t - t_B)$, centered at $x = 0.5$. A shock is clearly visible at $x = 0.5$. No oscillations develop for the case $N = 1001$, where $a \equiv b$ for all times.

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