RESTRICTION OF SQUARE INTEGRABLE REPRESENTATIONS, A PARTIAL SUMMARY.

JORGE A. VARGAS

ABSTRACT. In this note we present, a review of some aspects on the problem of restricting square integrable representations of a semisimple Lie group to reductive subgroups.

1. INTRODUCTION.

An important problem in representation theory of Lie groups is to understand "branching laws". That is, given a unitary representation, (π, V) of a Lie group G, to describe the restriction of π to a Lie subgroup H of G as direct integral of unitary irreducible representations of H. In formulae, is to write down as explicit as possible a unitary, bijective, linear, $H-$map

$$\pi_{|H} \to \int_{i \in \hat{H}} m(i)\ V_i\ d\mu(i).$$

In this note we will be concern in understanding "branching laws" under the setting:

(RDS)
G is a connected, semisimple and finite center Lie group;
H is a closed connected reductive subgroup of G; and
(π, V) is an irreducible square integrable representation of G.

Certainly, this problem has many aspects and has been studied by many authors. In particular, T. Kobayashi in [13] [11] [12] [15] has proven substantial results that apply to our particular problem. Moreover, Gross-Wallach in [5] have studied this problem, obtaining explicit results in case of quaternionic discrete series representations.

Whenever (G, H) is a generalized symmetric pair, Kobayashi has given necessary and sufficient conditions for the restriction of an irreducible square integrable representation to have an admissible restriction to H. In this note we study problem (RDS) under two different point of views. One is by means of the explicit realization of a square integrable representation as an eigenspace of the Casimir operator, i.e. Hotta's realization, c.f. [9] [18]. In joint work with Bent Orsted we obtained information on the discrete spectrum of $\pi_{|H}$. In order to describe the results we set up some notation. Throughout this note K is a fixed maximal compact subgroup of G such that $L := H \cap K$ is a maximal compact subgroup of H. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{s} = \mathfrak{h} + \mathfrak{q}$ the Cartan decomposition of the complexification, $\mathfrak{g}$, of the the

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Lie algebra of G and the respective $Ad(H)-$invariant decomposition associated to $\mathfrak{h}$. Henceforth, (π, V) , will denote a square integrable irreducible representation of G. For a proof of the next statement we refer to [18].

Theorem 1. *The discrete spectrum of* $\pi_{|H}$ *is contained in the discrete spectrum of*

$$\bigoplus_{m\geq 0} L^2(H \times_{H\cap K} (S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W)).$$

Here, $S^m(Z)$ denotes the $m^{th}-$symmetric power of the vector space Z, W is the lowest $K-$type of (π, V) and $L^2(...)$ indicates the space of square integrable sections of the homogeneous vector bundle

$$H \times_{H\cap K} Z \longrightarrow H/H\cap K$$

induced by a representation of $H \cap K$ on a vector space Z.
Theorem 1 is a consequence of the (2,2)-continuity of the Berezin transform and its generalizations by means of normal derivatives for the immersion $H/H\cap K \hookrightarrow G/K$. c.f. §3.
As an example, we consider $G = Spin(2n, 1), H = Spin(2k) \times Spin(2n-2k, 1), 1 \leq k < n$. Since L is the product of two groups, we may write the restriction to L of the lowest $K-$type of (π, V) as

$$\oplus_j W_j \otimes Z_j,\ W_j \in \widehat{Spin(2k)},\ Z_j \in \widehat{Spin(2n-2k)}.$$

Also, a discrete factor of $\pi_{|H}$ is of the form $X \otimes Z$ with X (resp. Z) an irreducible representation of $Spin(2k)$ (resp. Z an irreducible square integrable representation of $Spin(2n-2k, 1)$. Then, the highest weight of X is of the type $(r, 0, \ldots, 0)$ plus a weight of W_j and Z is such that some $Spin(2n-2k)$ type of Z is a Z_j. For a proof, c.f.[18].

The second point of view is joint work in progress with Michel Duflo. As suggested by Duflo, we consider the moment map

$$p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^\star$$

of the coadjoint orbit, Ω, of G determinate by the Harish-Chandra parameter associated to a given square integrable representation (π, V) c.f. §4.
In particular, we show,

Theorem 2. *For a generalized symmetric pair* (G, H) *, we have that*

$\pi_{|H}$ *is admissible if and only if* $p_{\mathfrak{h}}$ *is a proper map.*

Secondly, to (π, V) , Harish-Chandra has associated a system of positive roots Ψ in a compact Cartan subgroup of G contained in K c.f. §4. Let $\mathfrak{z}$ denote the center of $\mathfrak{k}$. Then, in [3] we construct an ideal $\mathfrak{k}_1(\Psi)$ of $\mathfrak{k}$ so that

Proposition 1. (π, V) *restricted to* $\mathfrak{k}_1(\Psi) + \mathfrak{z}$ *is admissible.*

For each simple group G and for every Ψ, in [3], we compute the subalgebra $\mathfrak{k}_1(\Psi)$. Finally, let β_Ω denote the Liouville measure on the orbit Ω. When $p_{\mathfrak{h}}$ is a proper map, and hence, $\pi_{|H}$ is admissible we obtain a relation between the multiplicity of the irreducible factors of $\pi_{|H}$ and the push-forward of the measure β_Ω, generalizing Theorems of Duflo-Heckman-Vergne for the case $H = K$ and results of Heckman in the case $G = K$ and $H = L$. c.f. [2], [8].

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2. Some examples on the problem of restricting discrete series.

Let us recall that a unitary representation $\pi_{|H}$ of a connected reductive group H is unitarily equivalent to a direct integral decomposition into irreducible unitary representations. That is, there exists a Borel measurable function $m : \hat{H} \to [0, +\infty]$ (the multiplicity function), a representative (π_i, V_i) of each $i \in \hat{H}$ and a Borel measure μ on $\hat{H}$ so that $\pi_{|H}$ is unitarily equivalent to the direct integral

$$\int_{i \in \hat{H}} m(i) V_i \, d\mu(i).$$

By definition, a *discrete factor* of $\pi_{|H}$ is $i \in \hat{H}$ so that $\mu(i) > 0$. In this case, we have that $m(i) = dim Hom_H(V_i, V)$. The *discrete spectrum* of $\pi_{|H}$ is the closure of the linear subspace of V defined by the image in V of the obvious map from

$$\oplus_{\{i : \mu(i) > 0\}} Hom_H(V_i, V) \otimes V_i.$$

A representation of H is *admissible* whenever it is equal to its discrete spectrum and the multiplicity of each irreducible factor is finite. In [13] it is shown that whenever (π, V) is a square integrable representation of G, then the discrete factors of $\pi_{|H}$ are again square integrable representations and the support of measure μ is contained in the set of tempered representations of H.

We now state a result on the existence of discrete spectrum, or, equivalently of discrete factors for $\pi_{|H}$. We denote by (τ, W) the lowest $K-$type, in the sense of Vogan or Schmid, [21], [7] of (π, V) . Let (σ, Z_0) denote an irreducible factor of the restriction of τ to L.

Theorem 3. *Assume there exists a square integrable and irreducible representation (ρ, Z) of H so that its lowest $L-$type is equivalent to (σ, Z_0). Then, (ρ, Z) is a discrete factor of $\pi_{|H}$.*

For a proof c.f. [19].
For example, when G/K is a Hermitian symmetric space, H so that H/L is a real form of G/K and (π, V) is a holomorphic or antiholomorphic representation of G, then the whole discrete spectrum of $\pi_{|H}$ is given by Theorem 3. Since, in this case we actually show that $\pi_{|H}$ is unitarily equivalent to $\simeq L^2(H \times_L \tau_{|L})$. For a proof, c.f.[20].

Kobayashi in [11] has shown that whenever $\pi_{|H}$ is admissible, then all the discrete factors of $\pi_{|H}$ have the same the associated variety. When we consider $G = Spin(2n, 1)$, $2 \leq n$, $H = Spin(2, 1)$ imbedded in the obvious way, Theorem 3 gives us examples of discrete factors of $\pi_{|H}$ which have distinct associated variety. c.f. [18].

Next, we show two results on multiplicities of discrete factors.

Proposition 2. *Let (π, V) be a square integrable irreducible representation of $Spin(2n, 1)$. Then,*

i) The multiplicity of any discrete factor of (π, V) restricted to $Spin(2k) \times Spin(2n-2k,1)$ is finite.
ii) If $k > 0$, the multiplicity of any irreducible factor of the restriction of (π, V) to $Spin(2n-2k,1)$, is infinite.

For a proof c.f. [18].

In order to continue, we fix compatible Iwasawa decompositions for $Spin(2n,1) = KAN$, $Spin(2n-1,1) = LAN_1$ and let M denote the centralizer of A in L. Since $L = Spin(2n-1), M = Spin(2n-2)$, we have that rank of L is equal to the rank of M. We fix compatible orders in a maximal torus of M (or L), and hence, we may and will associate to each irreducible representation μ_j of L an irreducible representation σ_j of M having the same highest weight.

Let (π, V) be an irreducible square integrable representation of $Spin(2n,1)$ and let (τ, W) denote its lowest $K-$type. We have,

Proposition 3. *We decompose $\tau_{|L} = \mu_1 \oplus \cdots \oplus \mu_r$ as a sum of irreducible representations of L. Then*

$$\pi_{|Spin(2n-1,1)} \simeq \sum_{j=1}^{r} \int_{\mathfrak{a}^\star} Ind_{MAN}^{Spin(2n-1,1-}(\sigma_j \otimes e^{i\nu} \otimes 1) m_j(\nu) d\nu.$$

Moreover, $0 \le m_j(\nu) < \infty$, for every j, ν.

For a proof c.f. [19]. We actually show that $\pi_{|H}$ is unitarily equivalent to a subrepresentation of $L^2(H \times_L \tau_{|L})$ and determine the image of the intertwining map.

3. Berezin transform and discrete spectrum.

Let G, H, K, L, (π, V) be as before. Thus, (π, V) is a square integrable irreducible representation of G. Let (τ, W) be the lowest $K-$type of (π, V) in the sense of Schmid or Vogan, c.f. [7], [21], [9] . After the work of Hotta and Paratharasathy (π, V) is unitarily equivalent to the representation $(R, H^2(G, \tau))$ constructed as follows. Let $\bar{\Omega}$ denote the Casimir operator acting on $L^2(G \times_K W)$ as an essentially self adjoint elliptic differential operator. Let Λ be the Harish-Chandra parameter of (π, V) and let ρ that corresponds to the Weyl chamber containing Λ. c.f. §4 . Let

$$H^2(G, \tau) = \{f \in L^2(G \times_K W) : \bar{\Omega}(f) = [(\Lambda, \Lambda) - (\rho, \rho)]f\}$$

and let R denote the right regular action of G on $L^2(G \times_K W)$. Then Hotta has shown that (π, V) is unitarily equivalent to $(R, H^2(G, \tau))$. Because of the regularity theorem for elliptic systems of equations, $H^2(G, \tau)$ consists of real analytic sections. Therefore, we may restrict an element in $H^2(G, \tau)$, as well the normal derivatives of an element in $H^2(G, \tau)$ to the submanifold H/L. Because of our choice of maximal compact subgroups, the homogeneous bundle $H \times_L \tau_{|L} \to H/L$ is a subbundle of $G \times_K W$ and, hence, the restriction of an element of $H^2(G, \tau)$ to H/L is a smooth section of the bundle $H \times_L \tau_{|L} \to H/L$.
For each $m \ge 0$, let

$$r_m : H^2(G, \tau) \to C^\infty(H \times_L (S^m(\mathfrak{q} \cap \mathfrak{s}) \times W))$$

denote the linear map that associates to each $f \in H^2(G,\tau)$ the function $r_m(f)$ on H which takes on values on $S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W$ defined by

$$r_m(f)(h) = \sum_{\alpha_1+\cdots+\alpha_n=m} \ell_{X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}}(f)(h).$$

Here, $X_1,\cdots,X_n$ is a basis of $\mathfrak{q}\cap\mathfrak{s}$ and ℓ denotes left infinitesimal differentiation after we symmetrize $X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}$. As pointed in the previous paragraph it readily follows that $r_m(f)$ is a smooth section of the bundle $H\times_L(S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W)$. Besides, r_m intertwines the respective representations of H. We have,

Theorem 4. *For every $m \geq 0$,*

i) $r_m(H^2(G,\tau))$ is contained in $L^2(H\times_L S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W)$.
ii) $r_m : H^2(G,\tau))\to L^2(H\times_L S^m(\mathfrak{q}\cap\mathfrak{s})\otimes W)$ is a continuous linear map.
iii) If (R,Z) is a discrete factor of $\pi_{|H}$, then there exists $k\geq 0$ so that $r_k(Z)$ is nonzero. Therefore, a discrete factor of $\pi_{|H}$ is a discrete factor of $L^2(H\times_L S^k(\mathfrak{q}\cap\mathfrak{s})\otimes W)$ for some k.
iv) If we further assume that (π,V) is integrable, then $r_m r_m^\star$ is (p,p)-continuous for every p in the interval $[1,+\infty)$.

For a proof c.f. [18].
The Berezin transform is the linear operator $r_0 r_0^\star$. We note that Item iv) generalizes results of G.K. Zhang [23].

4. Moment map and admissibility.

In this section we will denote the Lie algebra of a Lie group by the corresponding German lower case letter. To denote the complexification of a Lie algebra we add the subscript $\mathbb{C}$. We fix T a maximal torus in K. Since we are assuming that G admits square integrable representations, because of the work of Harish-Chandra, we have that T is a Cartan subgroup of G. Let $\Lambda\in i\mathfrak{t}^\star$ be the differential at the identity of a character of T. We will denote by $WL(T)\subset i\mathfrak{t}^\star$ the lattice of such $\Lambda's$. Let us recall that $\lambda\in\mathfrak{t}^\star$ is regular if its centralizer in G is T. We denote by $WL(T)_{reg}$ the regular elements in $WL(T)$. Let W denote the Weyl group of T in G, equivalently W is the quotient of the normalizer of T in K by T. The parameterization, due to Harish-Chandra, of the set of equivalence classes of irreducible square integrable representations of G is by means of the set of orbits of W in $WL(T)_{reg}$. Let $(...,...)$ the inner product on $i\mathfrak{t}^\star$ constructed from the Killing form of $\mathfrak{g}$. Let $\Phi=\Phi_c\cup\Phi_n$ denote the set of roots of $\mathfrak{t}_\mathbb{C}$, Φ_c the compact roots and the Φ_n noncompact roots. Let (G,H) be a generalized symmetric space and let σ denote the involution whose fixed point set is the Lie algebra of the group H. Thus, σ leaves $\mathfrak{k}$ as well as $\mathfrak{s}$ invariant and $\mathfrak{l}:=\{X\in\mathfrak{k}:\sigma X=X\}$ is a maximally compact embedded subalgebra of $\mathfrak{h}$. Following T. Kobayashi we choose a maximal torus T so that

$$\sigma(\mathfrak{t})=\mathfrak{t}.$$
$$\mathfrak{t}_-^\sigma=\{X\in\mathfrak{t}:\sigma X=-X\}.$$

is a maximally split Cartan subspace for the pair (K,L). Once and for all, we fix a system of positive roots $\Delta\subset\Phi_c$ such that $\{\alpha_{|\mathfrak{t}_-^\sigma}\neq 0\}$ is a system of positive roots for the restricted root system $(\mathfrak{k},\mathfrak{t}_-^\sigma)$. Thus, a cross-section for the set of orbits of W in $WL(T)_{reg}$ is given by

$$WL_+ := \{\Lambda\in WL(T) : (\Lambda,\alpha)>0,\ \forall\,\alpha\in\ \Delta\}.$$

For $\Lambda \in i\mathfrak{t}^\star$, let $\lambda := (-i)\Lambda \in \mathfrak{t}^\star$. As usual, we think of the elements of $\mathfrak{t}^\star$ as the linear functional on $\mathfrak{g}$ extended by zero on the orthogonal complement of $\mathfrak{t}$. Thus, we may consider

$$\Omega := Ad^\star(G) \cdot \lambda$$

the coadjoint orbit of λ. Let

$$p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^\star$$

denote the restriction map. Each $\Lambda \in WL_+$ determines a system of positive roots for Φ, namely,

$$\Psi := \{\alpha \in \Phi : (\Lambda, \alpha) > 0\}.$$

Hence, $\Delta \subset \Psi$. Let $\Psi_n := \Psi \cap \Phi_n$. Following T. Kobayashi we consider the proper cone

$$\mathbb{R}^+\Psi_n := \sum_{\beta \in \Psi_n} \mathbb{R}_{\geq 0}\beta.$$

In [3] we find a proof of:

Theorem 5. *Let (G, H) be a generalized symmetric pair and let Ω be a coadjoint orbit trough $\lambda \in \mathfrak{t}^\star$. Let $\Lambda := i\lambda$ and assume that Λ is dominant for Ψ. Then,*

$$p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^\star \text{ is a proper map if and only if } \mathbb{R}^+\Psi_n \cap i\mathfrak{t}^\sigma_- = \{0\}.$$

For a proof, c.f. [3].

Next, let (π_Λ, V_Λ) denote the irreducible square integrable representation of G attached by Harish-Chandra to $\Lambda \in WL_+$. We may show,

Theorem 6. *The restriction of (π_Λ, V_Λ) to H is an admissible representation if and only if $p_{\mathfrak{h}} : G \cdot \lambda \to \mathfrak{h}^\star$ is a proper map.*

Let us recall that an element X of $\mathfrak{g}^\star$ is *strongly elliptic* if its centralizer in G is a compact subgroup.

Theorem 7. *Let (G, H) be an arbitrary reductive pair in the sense of* [15]*. Then, whenever $p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^\star$ is a proper map the set $p_{\mathfrak{h}}(\Omega)$ is contained in the set of strongly elliptic elements of $\mathfrak{h}^\star$.*

For a proof c.f. [3].

We now consider $T_1 \subset T$ a maximal torus of L and a coadjoint orbit so that $p_{\mathfrak{h}} \to \mathfrak{h}^\star$ is a proper map. Thus, Theorem 7 shows that $p_{\mathfrak{h}}(\lambda) \neq 0$. We fix $C_{\lambda, \mathfrak{t}_1^\star, +}$ a closed Weyl chamber for the pair $(\mathfrak{l}, \mathfrak{t}_1)$ such that $p_{\mathfrak{h}}(\lambda)$ is dominant with respect to $C_{\lambda, \mathfrak{t}_1^\star, +}$. We have

Theorem 8. *There exists a unique open Weyl chamber $C_{\lambda,+}$ for $(\mathfrak{h}, \mathfrak{t}_1)$ so that*

i) $C_{\lambda,+}$ is contained in $C_{\lambda, \mathfrak{t}_1^\star, +}$
ii) $p_{\mathfrak{h}}(\Omega) \cap C_{\lambda, \mathfrak{t}_1^\star, +}$ is contained in the relative closure of $C_{\lambda,+}$ in $C_{\lambda, \mathfrak{t}_1^\star, +}$.

For a proof c.f. [3].

The second statement of Theorem 8 let us apply a Theorem of Weinstein [22] and obtain

Proposition 4. *When $p_{\mathfrak{h}}$ is a proper map, then $p_{\mathfrak{h}}(\Omega) \cap C_{\lambda, \mathfrak{t}_1^\star, +}$ is a convex set.*

Theorem 8 has the following nice consequence

Proposition 5. *Assume that (π_Λ, V_Λ) has an admissible restriction to H. Then, the Harish-Chandra parameters of the irreducible factors of $\pi_{|H}$ lie in a unique Weyl chamber for $(\mathfrak{h}, \mathfrak{t}_1)$.*

For a proof c.f. [3].
This proposition generalizes the following facts: If a holomorphic (quaternionic) discrete series representation has an admissible restriction to H, then, the $H-$irreducible factors are holomorphic (quaternionic) discrete series for H. For a proof of these two statements c.f. [17],[16].

5. The subgroup $\mathfrak{k}_1(\Psi)$.

Let $G, K, t, W, \Lambda, \lambda, \Psi$ be as in the previous section. Let $\mathfrak{k}$ denote the Lie algebra of K. We denote by $\mathfrak{u}_\Psi$ the linear subspace of $\mathfrak{s}$ spanned by the root vectors on the noncompact roots in Ψ. Thus, $[\mathfrak{u}_\Psi, \mathfrak{u}_\Psi]$ is a linear subspace of $\mathfrak{k}_\mathbb{C}$. It readily follows that the ideal of $\mathfrak{k}_\mathbb{C}$ spanned by $[\mathfrak{u}_\Psi, \mathfrak{u}_\Psi]$ is the complexification of an an ideal in $\mathfrak{k}$. From now on, we denote this ideal of $\mathfrak{k}$ by $\mathfrak{k}_1(\Psi)$.

In [3] we have computed the ideal $\mathfrak{k}_1(\Psi)$ for each simple real Lie algebra having a compactly embedded Cartan subalgebra. That is, when G has a nonempty discrete series.

Let $\mathfrak{z}$ denote the center of $\mathfrak{k}$. We fix $\Lambda \in WL_+$ and let Ψ the system of positive roots that makes Λ dominant.

Proposition 6. *Let $\lambda := -i\Lambda$ and $\Omega = Ad^\star(G) \cdot \lambda$. Then, the moment map (i.e the restriction map) $p : \Omega \to \mathfrak{k}_1(\Psi)^\star + \mathfrak{z}^\star$ is proper.*

For a proof c.f. [3]. Therefore, if (π_Λ, V_Λ) is the square integrable representation attached to Λ and H is so that $\mathfrak{k}_1(\Psi) + \mathfrak{z} \subseteq \mathfrak{h}$, then (π_Λ, V_Λ) restricted to H is admissible. In particular, whenever $\mathfrak{k}$ is a semisimple Lie algebra, we have that (π_Λ, V_Λ) has an admissible restriction to $\mathfrak{k}_1(\Psi)$.

Proposition 7. *We further assume that $L := K \cap H$ is normalized by T. Then, $\mathfrak{k}_1(\Psi) \subseteq \mathfrak{l}$ if and only if for each $w \in W$, $w\Psi_n \cup \Psi(\mathfrak{k}/\mathfrak{l})$ is contained in an open half space of $i\mathfrak{t}^\star$.*

Here, $\Phi(\mathfrak{r}/\mathfrak{p})$ is the set of roots of $\mathfrak{t}$ so that its root space is contained in the quotient of $\mathfrak{t}$ invariant subspaces $\mathfrak{p} \subseteq \mathfrak{r}$. This proposition allows us to prove an analogue of the theorems of Heckman and Duflo-Heckman-Vergne which determine the push-forward of the Liouville measure on Ω, c.f. [8], [2].

6. Push-forward of the Liouville measure.

We keep the notation of the last two sections. In order to simplify the exposition we further assume for this section that T is a maximal torus of L. Thus,

$$T \subset L \subset K.$$

Let β_Ω denote the Liouville measure on the coadjoint orbit $\Omega = Ad^\star(G) \cdot \lambda$. This is G invariant measure on Ω. For details c.f. [2] . From now on, we suppose that $p_\mathfrak{h}$ is a proper map. Therefore, $p_\mathfrak{h}(\Omega)$ is contained in the set of strongly elliptic elements of $\mathfrak{h}^\star$. Furthermore, Heckman in [8] has shown that either $p_\mathfrak{h}$ or $p_\mathfrak{l}$ are submersions on an open dense subset of Ω whose complement has Lebesgue measure zero. For a closed subgroup U of G so that $T \subset U$, we denote the Liouville measure of the coadjoint orbit $U \cdot \nu$, evaluated on a test function φ, by

$$\beta_{U\cdot\nu}(\varphi) := A_U(\varphi)(\nu) = \prod_{\alpha\in\Psi(\mathfrak{u})} (\alpha,\nu) \int_U \varphi(Ad(u)\cdot\nu)du.$$

Here, du is a conveniently chosen Haar measure on U. For the constants involved and proofs we refer to [1], Chapter VII.
Therefore, for $\mathfrak{u}$ equal to either $\mathfrak{h}$ or $\mathfrak{l}$, there exists functions

$$\Gamma_{\mathfrak{g},\mathfrak{u}} : \mathfrak{t}^\star \to \mathbb{R}$$

so that the push-forward, $(p_\mathfrak{u})_\star(\beta_\Omega)$ of the measure β_Ω by $p_\mathfrak{u}$ is given by

$$(p_\mathfrak{u})_\star(\beta_\Omega)(\varphi) := (p_\mathfrak{u})_\star(\beta_{G\cdot\lambda})(\varphi) = \int_{\mathfrak{t}^\star} \Gamma_{\mathfrak{g},\mathfrak{l}}(\mu) A_U(\varphi)(\mu)d\mu.$$

Here, $d\mu$ is a convenient Lebesgue measure on $\mathfrak{t}^\star$ and φ is a test function on $\mathfrak{t}^\star$. Moreover, $\Gamma_{\mathfrak{g},\mathfrak{u}}$ is a smooth function on the set of regular values of $p_\mathfrak{u}$. For each root $\beta \in \Phi(\mathfrak{g},\mathfrak{t})$ let $Y_{-i\beta}$ be the Heaviside generalized function on $\mathfrak{t}^\star$ associated to $-i\beta$. Thus,

$$Y_{-i\beta}(\varphi) = \int_0^\infty \varphi(t(-i\beta))dt.$$

We define, the convolution product of distributions,

$$Y_n^+ = \bigstar_{\beta\in\Psi_n} Y_{\beta/i}, \qquad Y_{\mathfrak{k},\mathfrak{l}}^+ = \bigstar_{\alpha\in\Psi(\mathfrak{k}/\mathfrak{l})} Y_{\alpha/i}.$$

In [6] we find a proof that our assumption on Ψ imply that the convolutions in the following statement are well defined.

Proposition 8.

$$\Gamma_{\mathfrak{g},\mathfrak{l}} = \sum_{w\in W} \epsilon(w)\delta_{w\lambda} \star w(Y_n^+) \star Y_{\mathfrak{k},\mathfrak{l}}^+.$$

$$\Gamma_{\mathfrak{g},\mathfrak{h}} = \sum_{w\in W} \epsilon(w)d(w)\delta_{w\lambda} \star [\bigstar_{\beta\in w\Psi_n\cap\Phi(\mathfrak{g}/\mathfrak{h})} Y_{-i\beta}] \star Y_{\mathfrak{k}/\mathfrak{l}}^+$$

Here, $d(w) = (-1)^{card[w\Psi_n\cap\Phi(\mathfrak{g}/\mathfrak{h})]}$.

Similarly to Lemma 6.1 [8] of Heckman we have that the functions $\Gamma_{\mathfrak{g},\mathfrak{l}}, \Gamma_{\mathfrak{g},\mathfrak{h}}$ in Proposition 8 are solutions to differential equations and that $\Gamma_{\mathfrak{g},\mathfrak{h}}$ is a derivative of $\Gamma_{\mathfrak{g},\mathfrak{l}}$. More precisely,

Proposition 9. $\Gamma_{\mathfrak{g},\mathfrak{l}}$ *is a solution to the equation in* $\theta \in \mathcal{D}'(i\mathfrak{t}^\star)$

$$\prod_{\alpha\in\Psi(\mathfrak{g}/\mathfrak{l})} \frac{\partial}{\partial(i\alpha)}\theta = \sum_{w\in W} \epsilon(w)\delta_{w\lambda}$$

whereas $\Gamma_{\mathfrak{g},\mathfrak{h}}$ *is a solution to the equation in* $\theta \in \mathcal{D}'(i\mathfrak{t}^\star)$

$$\prod_{\alpha\in\Psi(\mathfrak{g}/\mathfrak{h})} \frac{\partial}{\partial(i\alpha)}\theta = \sum_{w\in W} \epsilon(w)\delta_{w\lambda}$$

and

$$\Gamma_{\mathfrak{g},\mathfrak{h}} = \prod_{\alpha\in\Psi_n\cap\Psi(\mathfrak{h},\mathfrak{t})} \frac{\partial}{\partial(i\alpha)}\Gamma_{\mathfrak{g},\mathfrak{l}}.$$

The last statement reflects the fact that when we restrict discrete series representations and the restriction is admissible, then the multiplicities of the L irreducible factors determine the multiplicities of the H irreducible factors. For details and a proof c.f. [3].

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FAMAF-CIEM, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentine

E-mail address: vargas@famaf.unc.edu.ar