EXAMPLES ADMISSIBLE RESTRICTION TO L

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Abstract. In this note,

1. Admissible restriction to L which does not contain $K_1(\Psi)$

Example 1: Let $G = SO(2n,1)$ and L a copy of $U(n)$ contained in $SO(2n)$, then any discrete series for $SO(2n,1)$ has an admissible restriction to $U(n)$.

In order to show the statement we need some notation and facts. We fix $\mathfrak{t}$ a maximal torus in $\mathfrak{so}(2n)$. Hence, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{so}(2n,1)$ and there exists an orthogonal basis $\epsilon_1, \cdots, \epsilon_n$ of $i\mathfrak{t}^\star$ so that

$$\Phi_{\mathfrak{k}} = \{\pm\epsilon_i \pm \epsilon_j, i \neq j\}, \quad \Phi_n = \{\pm\epsilon_j\}.$$

Let Y_α be root vectors associated to a root α in such a way

$$\bar{Y_\alpha} = -Y_{-\alpha}, \alpha \in \Phi_{\mathfrak{k}},\ \bar{Y_\alpha} = Y_{-\alpha}, \alpha \in \Phi_n,\ [Y_\alpha, Y_{-\alpha}] = \frac{2}{(\alpha,\alpha)} H_\alpha.$$

Therefore, $\mathfrak{t}_{\mathbb{C}}$ together with $\{Y_\alpha, \alpha \in \{\pm(\epsilon_i - \epsilon_j)\}\}$ span a subalgebra which is the complexication of subalgebra $\mathfrak{l}$ isomorphic to $\mathfrak{u}(n)$.

Next, we describe the associated variety of an irreducible square integrable representation of $SO(2n,1)$. For this we recall $\mathfrak{p}_{\mathbb{C}} = \sum_{1\leq i\leq n} \mathbb{C}Y_{\pm\epsilon_i}$, the representation of $SO(2n)$ in $\mathfrak{p}_{\mathbb{C}}$ is the first fundamental representation and the Killing form is $B(\sum z_j Y_{\epsilon_j} + \sum w_k Y_{-\epsilon_k}, \sum z_j Y_{\epsilon_j} + \sum w_k Y_{-\epsilon_k}) = 2\sum z_j w_j$. The orbits of $SO(2n,\mathbb{C})$ in $\mathfrak{p}_{\mathbb{C}}$ are

$$\{B(Y,Y) = c,\}\ c \neq 0;\ \{B(Y,Y) = 0, Y \neq 0\};\ \{0\}$$

The nilpotent cone $\mathfrak{N}$ is $B(Y,Y) = 0$, its dimension is $2n-1$. In [?] we find a proof that the dimension of the associated variety of a discrete series for $SO(2n,1)$ is $2n-1$, hence, the associated variety of a discrete series for $SO(2n,1)$ is $\mathfrak{N}$. [?] has shown that the associated variety of a discrete series is $Ass(\pi) := Ad(K_{\mathbb{C}})(\sum_{-\beta\in\Psi_n} \mathbb{C}Y_\beta)$ and the ideal which defines the associated variety is a radical ideal. Another piece of information we need is a Theorem of Huang-Vogan [?] which shows that for a compact subgroup L of K, a discrete series π has an admissible restriction to L if and only if $\mathbb{C}[Ass(\pi)]$ is an admissible $L-$module, in turn, because of the Hilbert-Godement Theorem [?], and its generalization to affine irreducible varieties due to Vergne, Knopp, this is equivalent to $\mathbb{C}[Ass(\pi)]^L = \mathbb{C}$. We now verify $\mathbb{C}[Ass(\pi)]^{U(n)} = \mathbb{C}$. Let V denote the first fundamental representation of $U(n)$, then as a representation for $U(n)$ $\mathfrak{p}_{\mathbb{C}} = V \oplus V^\star$. Here, $V = \sum \mathbb{C}Y_{\epsilon_j}$, and by mean of the Killing form $V^\star = \sum \mathbb{C}Y_{-\epsilon_j}$. Let $e(v+\lambda) = \lambda(v), v \in V, \lambda \in V^\star$. Then e is a polynomial function on $\mathfrak{p}_{\mathbb{C}}$ which is $U(n)-$invariant. We notice that $e(\mathfrak{N}) = 0$. Besides, owing to classical invariant theory we have that $\mathbb{C}[\mathfrak{p}_{\mathbb{C}}]^{U(n)} = \mathbb{C}[e]$. Thus, if $p \in \mathbb{C}[As(\pi)]^{U(n)}$, choose $q \in \mathbb{C}[\mathfrak{p}_{\mathbb{C}}]^{U(n)}$ so that $q_{|\mathfrak{N}} = p$. Thus, p is constant and we have shown that π restricted to $U(n)$ is admissible. We would like to point out that the same statement follows if we apply Theorem..$As(\pi) \cap i\mathfrak{t}^\star_{-} = 0$....in Kob or from the fact that

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in this case $\mathbb{C}[\mathfrak{N}] = Ind_{SO(2n-1)}^{SO(2n)}(Trivial\ rep.) = \sum_k V_{k\epsilon_1}$, next we apply the Mackey restriction Theorem and finally the Theorem of Huang Vogan cited previously.

Example 2: Let $Spin(9) \to SO(\mathbb{R}^{16})$ denote the spin representation, we identify $Spin(9)$ with its image in $SO(16)$. We claim that a discrete series for $SO(16,1)$ has an admissible restriction to $Spin(9)$. The same result holds for the spin representation $Spin(7) \to SO(8)$ and a discrete series for $SO(8,1)$.

In order to verify the statement, we recall that for both cases, $\mathbb{C}^\star \times Spin(2k+1)$ acting on $\mathbb{C}^{2^k}$ is a multiplicity free representation. We now recall the structure of $\mathbb{C}[\mathbb{C}^{2^k}]$ as a $\mathbb{C}^\star \times Spin(2k+1)$ module. For $\mathbb{C}^\star \times Spin(7)$ the highest weight vectors of an irreducible subrepresentation of $\mathbb{C}[\mathbb{C}^8]$ are the polynomials $h_1^{k_1} h_2^{k_2}, k_j \geq 0$ where h_1 is a highest weight vector for the spin representation of $Spin(7)$, h_2 is a nonzero invariant quadratic form on $\mathbb{C}^8$. Since both h_1, h_2 are irreducible polynomials, we obtain that $\mathbb{C}[\mathfrak{N}] = \sum_{k \geq 0} V_{\frac{k}{2}(\epsilon_1+\epsilon_2+\epsilon_3)}$ where $V_{\frac{k}{2}(\epsilon_1+\epsilon_2+\epsilon_3)}$ denotes the irreducible representation of $Spin(7)$ of highest weight $\frac{k}{2}(\epsilon_1+\epsilon_2+\epsilon_3)$. Therefore, the action of $Spin(7)$ in $\mathbb{C}[\mathfrak{N}]$ is admissible. The theorem of Huang and Vogan lead us to the admissibility statement.

For the case $Spin(9) \to SO(\mathbb{R}^{16})$ the highest weight vectors of an irreducible subrepresentation of $\mathbb{C}[\mathbb{C}^{16}]$ are the polynomials $h_1^{k_1} h_2^{k_2} h_3^{k_3}, k_j \geq 0$ where h_1 is a highest weight vector for the spin representation of $Spin(9)$, h_2 is a highest weight vector for the first fundamental representation in $\mathfrak{C}^9$, degree of h_2 is two, h_3 is a nonzero invariant quadratic form on $\mathbb{C}^{16}$. Since h_1, h_2, h_3 are irreducible polynomials, we obtain that $\mathbb{C}[\mathfrak{N}] = \sum_{k_j \geq 0} V_{\frac{k_1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)} \oplus V_{k_2\epsilon_1}$ where V_γ denotes the irreducible representation of $Spin(9)$ of highest weight γ. Therefore, $\mathbb{C}[\mathfrak{N}]$ is admissible as $Spin(9)-$module.

Example 3: Let M be a complex connected reductive Lie group and $\rho : M \to GL(\mathbb{C}^n)$ a prehomogeneous space. This is equivalent to M has a dense orbit in $\mathbb{C}^n$. Thus, $\mathbb{C}[\mathbb{C}^n]^M = \mathbb{C}$. Next, we consider the group $SU(n,1)$ and a holomorphic discrete series (π_Λ, V_Λ) for $SU(n,1)$. A Cartan decomposition for $SU(n,1)$ is

$$\mathfrak{su}(n,1) = \mathfrak{k} \oplus (\mathbb{C}^n \oplus (\mathbb{C}^n)^\star).$$

Let $\tilde{L}$ denote a compact real form of $\rho(M)$ so that $\tilde{L} \subset U(n)$ and set $L \subset K$ the inverse image of $\tilde{L}$ by "half of the isotropy representation" We claim,

$$\pi_\Lambda \text{ restricted to } L \text{ is admissible.}$$

In fact, we may arrange matters so that the associated variety of π_Λ is equal to $\mathbb{C}^n$. Since $\mathbb{C}[\mathbb{C}^n]^M = \mathbb{C}$ a theorem of Huang and Vogan forces that π_Λ restricted to L is admissible.

We would like to point out that whenever G has a center of positive dimension, this example follows from the fact that holomorphic discrete series has an admissible restriction to the center of K and that then the center of K is contained in L. However, there are many examples prehomogeneous spaces where M is a semisimple Lie group. Some of them includes triples (ρ, M, V) such that the action of M on the symmetric functions of V is multiplicity free. Sato and Kimura have written the list of prehomogeneous (ρ, M, V) where G is semisimple and V is an irreducible representation. The list is:

(The list has repetitions because, some of them are multiplicity free space and others are not.)

$SL_n \times SL_m, \mathbb{C}^n \otimes \mathbb{C}^m, \frac{m}{2} \geq n \geq 1$, Mult. free are $SL_1 \times SL_m$

$SL_{2m+1}, \Lambda^2(\mathbb{C}^{2m+1})$ This is mult. free

$SL_{2n+1} \times SL_2, \Lambda^2(\mathbb{C}^{2n+1}) \otimes \mathbb{C}^2$,

$Sp(n) \times SL_{2m+1}, \mathbb{C}^{2n} \otimes \mathbb{C}^{2m+1}, n \geq 2m+1$ Mult. free is $Sp(n) \times SL_1$.

$Spin(10)$, half spin rep in $\mathbb{C}^{16}$ Multiplicity free

$Sp(2) \times SL_m, m \geq 5$ Multiplicity free

$G \times SL_m, \mathbb{C}^n \otimes \mathbb{C}^m, m \geq n \geq 1, G$ semismple and irreducible action on $\mathbb{C}^n$.

Example 4: This example is a byproduct of example 3. Let $M_j \subset GL(\mathbb{C}^{n_j}, j = 1, \cdots, k$ be reductive subgroups so that $\mathbb{C}[\mathbb{C}^{n_j}]^{M_j} = \mathbb{C}$. Set $M = M_1 \times \cdots M_k$. From this M construct L as in example 3. Then a holomorphic discrete series for $SU(n,1)$ has an admissible restriction to L. This follows from the equality

$$\mathbb{C}[\mathbb{C}^n]^M = \mathbb{C}[\mathbb{C}^{n_1}]^{M_1} \otimes \cdots \otimes \mathbb{C}[\mathbb{C}^{n_k}]^{M_k}$$

Question for Rubenthaler or...???

What are the reductive subgroups $M \subset GL(n, \mathbb{C})$ so that
— M leaves invariant a nondegenerate quadratic form b
— $\mathbb{C}[\mathbb{C}^n]^M = \mathbb{C}[b]$.

From the list of prehomogeneous spaces of Kimura and Sato Kimura I have extracted three examples of this question, they are the ones that we consider en example 1 and example 2.

A discrete series for $SO(2n,1)$ has an admissible restriction to the compact real form of such an M, pulled back to $SO(2n)$

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