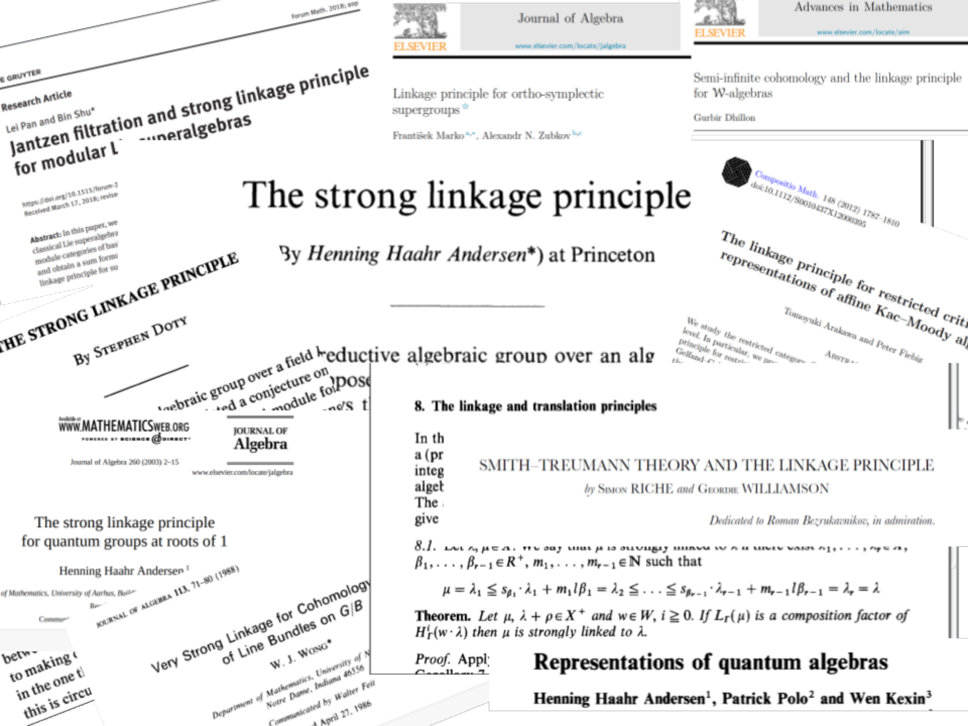
Linkage principle for small quantum groups

Cristian Vay

FaMAF – UNC, CIEM – CONICET

Linkage principle


Research Article
Lei Pan and Bin Shu*
Jantzen filtration and strong linkage principle for modular Lie superalgebras
Journal of Algebra
www.elsevier.com/locate/jalgebra
Linkage principle for ortho-symplectic supergroups
František Marko, Alexandr N. Zubkov
Advances in Mathematics
www.elsevier.com/locate/aim
Semi-infinite cohomology and the linkage principle for W-algebras
Gurbir Dhillon
The strong linkage principle
By Henning Haahr Andersen*) at Princeton
Compositio Math. 148 (2012) 1787–1810
The linkage principle for restricted critical level representations of affine Kac–Moody algebras
Tomoyuki Arakawa and Peter Fiebig
THE STRONG LINKAGE PRINCIPLE
By STEPHEN DOTY
8. The linkage and translation principles
SMITH–TREUMANN THEORY AND THE LINKAGE PRINCIPLE
by SIMON RICHE and GEORDIE WILLIAMSON
Dedicated to Roman Bezrukavnikov, in admiration.
WWW.MATHEMATICSWEB.ORG
JOURNAL OF Algebra
Journal of Algebra 260 (2003) 2–15
www.elsevier.com/locate/jalgebra
The strong linkage principle for quantum groups at roots of 1
Henning Haahr Andersen
$\beta_1, \ldots, \beta_{r-1} \in R^+$, $m_1, \ldots, m_{r-1} \in \mathbb{N}$ such that
$\mu = \lambda_1 \leq s_{\beta_1} \cdot \lambda_1 + m_1 l \beta_1 = \lambda_2 \leq \ldots \leq s_{\beta_{r-1}} \cdot \lambda_{r-1} + m_{r-1} l \beta_{r-1} = \lambda_r = \lambda$
Theorem. Let $\mu, \lambda + \rho \in X^+$ and $w \in W$, $i \geq 0$. If $L_r(\mu)$ is a composition factor of $H^i_r(w \cdot \lambda)$ then μ is strongly linked to λ.
JOURNAL OF ALGEBRA 113, 71–80 (1988)
Very Strong Linkage for Cohomology of Line Bundles on G/B
W. J. WONG*
Communicated by Walter Feit
Representations of quantum algebras
Henning Haahr Andersen, Patrick Polo and Wen Kexin

Slogan

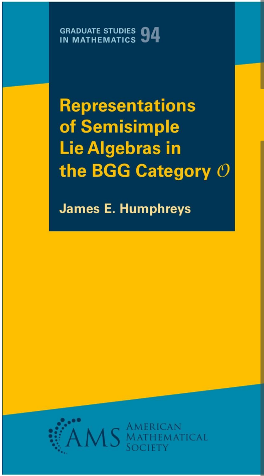
The linkage principle characterizes the composition factors of the Verma modules

GRADUATE STUDIES IN MATHEMATICS 94

Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$

James E. Humphreys

AMS American Mathematical Society



Definition. For $w \in W$ and $\lambda \in \mathfrak{h}^*$, define a shifted action of W (called the **dot action**) by $w \cdot \lambda = w(\lambda + \rho) - \rho$. If $\lambda, \mu \in \mathfrak{h}^*$, we say that λ and μ are **linked** (or W-linked) if for some $w \in W$, we have $\mu = w \cdot \lambda$. Linkage is clearly an equivalence relation on $\mathfrak{h}^*$. The orbit $\{w \cdot \lambda \mid w \in W\}$ of λ under the dot action is called the **linkage class** (or W-linkage class) of λ.

Working in $\mathcal{X}_0$ does make it possible to reformulate the problem in a useful way, taking advantage of Harish-Chandra's Theorem: first write

(1) $$\operatorname{ch} M(\lambda) = \sum_{\mu} a(\lambda, \mu) \operatorname{ch} L(\mu) \text{ with } a(\lambda, \mu) \in \mathbb{Z}^+ \text{ and } a(\lambda, \lambda) = 1.$$

Here μ ranges over weights $\leq \lambda$ and linked to λ by W, while $a(\lambda, \mu) = [M(\lambda) : L(\mu)]$. The partial ordering permits us to invert the resulting triangular system of equations:

(2) $$\operatorname{ch} L(\lambda) = \sum_{\mu} b(\lambda, \mu) \operatorname{ch} M(\mu) \text{ with } b(\lambda, \mu) \in \mathbb{Z} \text{ and } b(\lambda, \lambda) = 1.$$

The sum is again taken over weights $\mu \leq \lambda$ linked to λ. To make the role of W more explicit we can recast this as:

(3) $$\operatorname{ch} L(\lambda) = \sum_{w \cdot \lambda \leq \lambda} b(\lambda, w) \operatorname{ch} M(w \cdot \lambda) \text{ with } b(\lambda, w) \in \mathbb{Z},\ b(\lambda, 1) = 1.$$

This last format turns out to be most useful in practice, though the determination of the coefficients is typically quite subtle. Even in the case $\lambda = 0$, where the left side is known to be equal to $e(0)$, it is nontrivial to fill in the right side.

Small quantum groups

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 1, January 1990

FINITE DIMENSIONAL HOPF ALGEBRAS ARISING FROM QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS

GEORGE LUSZTIG

INTRODUCTION

0.1. An important role in the theory of modular representations is played by certain finite dimensional Hopf algebras $\bar{\mathfrak{u}}$ over F_p (the field with p elements, $p = \text{prime}$). Originally, $\bar{\mathfrak{u}}$ was defined (Curtis [3]) as the restricted enveloping algebra of a "simple" Lie algebra over F_p.

For our purposes, it will be more convenient to define $\bar{\mathfrak{u}}$ as follows.

Let us fix an indecomposable positive definite symmetric Cartan matrix

(a) $$(a_{ij})_{1\le i,j\le n}.$$

In particular $a_{ii} = 2$ and $a_{ij} = a_{ji} \in \{0, -1\}$, for $i \neq j$. Let $\overline{U}_Q$ be the Q-algebra defined by the generators $\overline{E}_i$, $\overline{F}_i$, H_i $(1 \le i \le n)$, and the relations

(b1) $$H_iH_j = H_jH_i,$$

(b2) $$H_i\overline{E}_j - \overline{E}_jH_i = a_{ij}\overline{E}_j, \qquad H_i\overline{F}_j - \overline{F}_jH_i = -a_{ij}\overline{F}_j,$$

(b3) $$\overline{E}_i\overline{F}_j - \overline{F}_j\overline{E}_i = \delta_{ij}H_i,$$

(b4) $$\overline{E}_i\overline{E}_j = \overline{E}_j\overline{E}_i, \quad \overline{F}_i\overline{F}_j = \overline{F}_j\overline{F}_i, \quad \text{if } a_{ij} = 0,$$

(b5)
$$\overline{E}_i^2\overline{E}_j - 2\overline{E}_i\overline{E}_j\overline{E}_i + \overline{E}_j\overline{E}_i^2 = 0, \quad \overline{F}_i^2\overline{F}_j - 2\overline{F}_i\overline{F}_j\overline{F}_i + \overline{F}_j\overline{F}_i^2 = 0, \quad \text{if } a_{ij} = -1.$$

Then $\overline{U}_Q$ is known to be the enveloping algebra of the simple Lie algebra $\mathfrak{g}$ over Q corresponding to (a).

Chevalley [2] has proved that any $\overline{U}_Q$-module of finite dimension over Q

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For our purposes, it will be more convenient to define $\overline{\mathfrak{u}}$ as follows.

Let us fix an i

(a)

In particular a_{ii}

Q-algebra define

0.3. Let $\mathscr{B}$ be the ring $\mathbb{Z}/[\zeta]$, where ζ is a pth root of 1, $\zeta \neq 1$, p an odd prime; thus $\mathscr{B}$ is the ring of integers in a cyclotomic filed $\mathscr{B}'$. One of the main results of this paper is that $\overline{\mathfrak{u}}$ can be regarded naturally as reduction modulo a maximal ideal of a Hopf algebra over $\mathscr{B}$. More precisely, we shall define a Hopf algebra $\tilde{\mathfrak{u}}$ over $\mathscr{B}$ with the following properties.

(b1)

(b2) $H_i\overline{E}_j - \overline{E}_jH_i = a_{ij}\overline{E}_j, \qquad H_i\overline{F}_j - \overline{F}_jH_i = -a_{ij}\overline{F}_j,$

(b3)

(b4)

(b5)

$\overline{E}_i^2\overline{E}_j - 2\overline{E}_i\overline{E}_j\overline{E}_i$

In particular, the simple $'\tilde{\mathfrak{u}}$-modules are in natural bijection with the simple $\overline{\mathfrak{u}}$-modules so that the simple $\overline{\mathfrak{u}}$-module corresponding to a simple $\tilde{\mathfrak{u}}$-module M has dimension $\leq \dim M$.

We conjecture that the last inequality is an equality (at least for p not too small) and that in fact $\overline{\mathfrak{u}}$ and $'\tilde{\mathfrak{u}}$ have identical representation theories.

Then $\overline{U}_Q$ is k

over Q corresponding to (a).

220 ASTÉRISQUE

1994

REPRESENTATIONS OF QUANTUM GROUPS AT A *p-TH* ROOT OF UNITY AND OF SEMISIMPLE GROUPS IN CHARACTERISTIC *p*: INDEPENDENCE OF *p*

H.H. ANDERSEN, J.C. JANTZEN, W. SOERGEL

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

220 ASTÉRISQUE

REPRESENTATI
QUANTUM GROUPS
ROOT OF UN
AND OF SEMISIMPLE
CHARACTERISTIC p: INDE

The representation theory of the $U^{[p]}(\mathfrak{g}_k)$ turns out to have many features that are (conjecturally) independent of p. Let us mention first the one most easily described. Since $U^{[p]}(\mathfrak{g}_k)$ has finite dimension, it is the direct product of indecomposable algebras, the blocks of $U^{[p]}(\mathfrak{g}_k)$. Each indecomposable restricted $\mathfrak{g}_k$–module M belongs to exactly one of these blocks; it is the unique block not annihilating M. Denote by $\mathcal{B}_k$ the block of the trivial one dimensional $\mathfrak{g}_k$–module. Work ([Hu2]) by Humphreys from 1971 showed: If p is greater than the Coxeter number h of R, then the simple modules belonging to $\mathcal{B}_k$ are indexed by the Weyl group W. The Cartan matrix of $\mathcal{B}_k$ is therefore a $(W \times W)$–matrix. In the cases known at that time (and in the cases known today) this matrix is independent of p (as long as $p > h$). So one might conjecture that this independence should hold in general. (This conjecture is implicitly contained in Verma's last conjecture in [Ver] to be discussed below.) We shall prove:

Theorem 1: *There is a* $\mathbf{Z}$*–algebra* $\mathcal{B}$ *(finitely generated as a* $\mathbf{Z}$*–module) such that for all* k *with* $\mathrm{char}(k) > h$ *the block* $\mathcal{B}_k$ *is Morita equivalent to* $\mathcal{B} \otimes_{\mathbf{Z}} k$.

H.H. ANDERSEN, J.C. JANTZEN, W. SOERGEL

The algebra $\mathcal{B}$ has also an interpretation in characteristic 0. Take an odd integer $p > 1$ (prime to 3 if R has a G_2 component) and consider the quantized enveloping algebra U_p at a p–th root of unity. Here we take Lusztig's version constructed via divided powers. It contains a finite dimensional analogue $\mathbf{u}_p$ of the restricted enveloping algebra. (This was discovered by Lusztig, cf. [Lu6] and [Lu7].) Then:

Theorem 2: *If* $p > h$*, then* $\mathcal{B} \otimes_{\mathbf{Z}} \mathbf{Q}(\sqrt[p]{1})$ *is Morita equivalent to the block of* $\mathbf{u}_p$ *containing the trivial one dimensional module.*

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Nichols algebras of diagonal type

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Advances in Mathematics 220 (2009) 59–124

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Classification of arithmetic root systems

I. Heckenberger

Universität München, Theresienstraße 39, 80333 Munich, Germany

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Communicated by Andrei Zelevinsky

$\Delta_+^{\chi} = \{\beta_1, \dots, \beta_t\}$

Abstract

Arithmetic root systems are invariants of Nichols algebras of diagonal type with a certain finiteness property. They can also be considered as generalizations of ordinary root systems with rich structure and many new examples. On the other hand, Nichols algebras are fundamental objects in the construction of quantized enveloping algebras, in the noncommutative differential geometry of quantum groups, and in the classification of pointed Hopf algebras by the lifting method of Andruskiewitsch and Schneider. In the present paper arithmetic root systems are classified in full generality. As a byproduct many new finite dimensional pointed Hopf algebras are obtained.

$\{E_{\beta_1}^{n_1} \cdots \cdots E_{\beta_t}^{n_t} \mid 0 \le n_i < N_i\}$ PBW basis of The Nichols alg $\mathcal{B}_\chi$

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On finite dimensional Nichols algebras of diagonal type

Nicolás Andruskiewitsch[1] · Iván Angiono[1]

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Abstract This is a survey on Nichols algebras of diagonal type with finite dimension, or more generally with arithmetic root system. The knowledge of these algebras is the cornerstone of the classification program of pointed Hopf algebras with finite dimension, or finite Gelfand–Kirillov dimension; and their structure should be indispensable for the understanding of the representation theory, the computation of the various cohomologies, and many other aspects of finite dimensional pointed Hopf algebras. These Nichols algebras were classified in Heckenberger (Adv Math 220:59–124, 2009) as a notable application of the notions of Weyl groupoid and generalized root system (Heckenberger in Invent Math 164:175–188, 2006; Heckenberger and Yamane in Math Z 259:255–276, 2008). In the first part of this monograph, we give an overview of the theory of Nichols algebras of diagonal type. This includes a discussion of the notion of generalized root system and its appearance in the contexts of Nichols algebras of diagonal type and (modular) Lie superalgebras. In the second and third part, we describe for each Nichols algebra in the list of Heckenberger (2009)

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On finite dimensional Nichols algebras of diagonal type

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7. Assume now that $\Bbbk$ is algebraically closed and of characteristic 0. The classification of the braided vector spaces (V, c) of diagonal type with finite-dimensional $\mathcal{B}(V)$ was obtained in [46]. (When char $\Bbbk > 0$, the classification is known under the hypothesis $\dim V \leq 3$ [51,85]). The core of the approach is the notion of generalized root system; actually, the paper [46] contains the list of all (V, c) of diagonal type with connected Dynkin diagram and finite generalized root system (these are called *arithmetic*). The list can be roughly split in several classes:

- Standard type [2], that includes Cartan type [19]; related to the Lie algebras in the Killing–Cartan classification.
- Super type [10], related to the finite-dimensional contragredient Lie superalgebras in characteristic 0, classified in [57].
- Modular type [3], related to the finite-dimensional contragredient Lie (super) algebras in positive characteristic, classified in [29,59].
- A short list of examples not (yet) related to Lie theory, baptised *UFO*'s.

The goal of this work is to give exhaustive information on the structure of these Nichols algebras.

8. This monograph has three Parts. Part I is an exposition of the basics of Nichols

Small quantum groups associatted to generalized root systems

$\mathfrak{q} \in \mathbb{C}^{\theta \times \theta}$

$\rightsquigarrow$

$\mathfrak{B}_{\mathfrak{q}}$ finite-dimensional Nichols algebras of diagonal type

$\rightsquigarrow$

$\mathfrak{B}_{\mathfrak{q}} \# \mathbb{C}\Gamma$ bosonization over an abelian group

$\rightsquigarrow$

$U_{\mathfrak{q}} = \mathcal{D}(\mathfrak{B}_{\mathfrak{q}} \# \mathbb{C}\Gamma)$ Drinfeld double

Small quantum group

Ex $\mathfrak{g}$ = Lie alg with Cartan matrix C

$\mathfrak{q} = (q^{c_{ij}})$

$U_q \twoheadrightarrow u_q$

Main results

Triangular decomposition

$\Bbbk\mathbb{Z}^\theta \times \mathbb{Z}^\theta = \Bbbk\langle K_\alpha, L_\beta \mid \alpha, \beta \in \mathbb{Z}^\theta \rangle$

$\mathbb{Z}^\theta$-graded

B_q

$$U_{\mathfrak{q}} = U_{\mathfrak{q}}^{-} \otimes U_{\mathfrak{q}}^{0} \otimes U_{\mathfrak{q}}^{+}$$

Rmk

$\mu \in \mathbb{Z}^\theta$, $u \in (U_q)_\mu$, $s \in U_q^0 \Rightarrow su = u\tilde{\mu}(s)$

where $\tilde{\mu}(K_\alpha L_\beta) = \frac{f(\alpha, \mu)}{f(\mu, \beta)} K_\alpha L_\beta$

Verma modules

Fix $\pi : U^0_{\mathfrak{f}} \longrightarrow \mathbb{k}$

$$Z_{\mathbb{k}}(\mu) = U_{\mathfrak{q}} \otimes_{U^0_{\mathfrak{q}} U^+_{\mathfrak{q}}} \mathbb{k}^{\mu}$$

Rmk

$$\operatorname{ch} Z_{\mathbb{k}}(\mu) = \prod_{\beta \in \Delta^{+}_{+}} \frac{1 - e^{-b^{\chi}(\beta)\beta}}{1 - e^{\beta}}$$

$\mathbb{k}|\mu\rangle$ with deg $\mu \in \mathbb{Z}^{\theta}$

$S(u) \cdot |\mu\rangle = \varepsilon(u)\, \pi^{\tilde{\mu}}(S)\, |\mu\rangle$

Simple modules

$$L_{\Bbbk}(\mu) = \operatorname{head} Z_{\Bbbk}(\mu)$$

Definition

Given $\beta \in \Delta_+^{\mathfrak{q}}$, $\mu \in \mathbb{Z}^\theta$, let $n \in \mathbb{N}$ be minimum such that

$$\mathfrak{q}(\beta,\beta)^n - \rho^{\mathfrak{q}}(\beta)\,\pi\widetilde{\mu}(K_\beta L_\beta^{-1}) = 0$$

$\rho : \mathbb{Z}^\theta \to \mathbb{k}^\times$, $\alpha_i \to \rho(\alpha_i,\alpha_i)$

and set

$$n_\beta^\pi(\mu) = \begin{cases} n & \text{if } 1 \le n \le b^{\mathfrak{q}}(\beta) - 1, \\ 0 & \text{if } n = b^{\mathfrak{q}}(\beta) \text{ or it does not exist.} \end{cases}$$

$\operatorname{ord} q(\beta,\beta)$

We define

$$\beta \downarrow \mu = \mu - n_\beta^\pi(\mu)\,\beta$$

Theorem

If $L_{\Bbbk}(\lambda)$ is a composition factor of $Z_{\Bbbk}(\mu)$, then $\lambda = \mu$ or there exist $\beta_1, ..., \beta_r \in \Delta_+^{\mathfrak{q}}$ such that

$$\lambda = \beta_r \downarrow \cdots \beta_1 \downarrow \mu$$

and $\mu - \beta_{top}^{\mathfrak{q}} \leq \lambda \leq \mu$

$$\hookrightarrow \sum_{\beta \in \Delta_+^{\mathfrak{q}}} \left(\mathrm{ht}^{\mathfrak{q}}(\beta) - 1\right)\beta$$

Corollary

The partion of $\mathbb{Z}^\theta$ given by $\downarrow$ determines the blocks of the category formed by the $\mathbb{Z}^\theta$-graded $U \otimes \mathbb{k}$-modules satisfying

Cat $\mathcal{C}_{\mathbb{k}}$

- M is finitely generated over $\mathbb{k}$;
- $M_\mu \mathbb{k} \subset M_\mu$ for all $\mu \in \mathbb{Z}^\theta$;
- $U_\nu M_\mu \subset M_{\nu+\mu}$ for all $\nu, \mu \in \mathbb{Z}^\theta$;
- $s\, m = m\, \pi(\widetilde{\mu}(s))$ for all $\mu \in \mathbb{Z}^\theta$, $m \in M_\mu$ and $s \in U^0$.

Projective simple modules

For $\beta \in \Delta_+^{\mathfrak{q}}$ and $\mu \in \mathbb{Z}^\theta$, let

$$\mathfrak{P}_{\Bbbk}^{\mathfrak{q}}(\mu) = \prod_{\substack{\beta\in\Delta_+^{\mathfrak{q}} \\ 1\le t<b^{\mathfrak{q}}(\beta)}} \prod_{1\le t<b^{\mathfrak{q}}(\beta)} \Big(\mathfrak{q}(\beta,\beta)^t - \rho^{\mathfrak{q}}(\beta)\,\pi\widetilde{\mu}(K_\beta L_\beta^{-1})\Big)$$

Corollary [AJS, Heckenberger-Yamane]

Let $\mu \in \mathbb{Z}^\theta$. The following are equivalent:

1. $\mathfrak{P}_{\Bbbk}^{\mathfrak{q}}(\mu) \neq 0$.
2. $Z_{\Bbbk}(\mu) = L_{\Bbbk}(\mu)$ is simple.
3. $Z_{\Bbbk}(\mu) = L_{\Bbbk}(\mu)$ is projective.

Corollary
If there is only one $\beta \in \Delta_+^{\mathfrak{q}}$ such that $\mathfrak{P}_{\Bbbk}^{\mathfrak{q}}(\beta, \mu) = 0$, then we have an exact sequence

$$0 \longrightarrow L_{\Bbbk}(\beta \downarrow \mu) \longrightarrow Z_{\Bbbk}(\mu) \longrightarrow L_{\Bbbk}(\mu) \longrightarrow 0.$$

and hence

$$\operatorname{ch} L_{\Bbbk}(\mu) = e^{\mu} \quad \frac{1 - e^{-n_{\beta}^{\pi}(\mu)\beta}}{1 - e^{-\beta}} \prod_{\gamma \in \Delta_+^{\mathfrak{q}} \setminus \{\beta\}} \frac{1 - e^{-b^{\mathfrak{q}}(\gamma)\gamma}}{1 - e^{-\gamma}}.$$

The linkage principle as a dot action

Let $\mathcal{W}^{\mathfrak{q}}_{\text{link}}$ be the group generated by all the affine reflections:

$$s_{\beta,m} \bullet \mu = s_\beta(\mu + m b^{\mathfrak{q}}(\beta)\beta - \varrho^{\mathfrak{q}}) + \varrho^{\mathfrak{q}}$$

$$\frac{1}{2}\sum_{\beta \in \Delta^{\mathfrak{q}}_{+}} (b^{\mathfrak{q}}(\beta) - 1)\beta$$

$\beta \in \Delta^{\mathfrak{q}}_{+,\text{car}}$, $m \in \mathbb{Z}$.

Corollary
Assume π is the trivial algebra map. If $L_{\Bbbk}(\lambda)$ is a composition factor of $Z_{\Bbbk}(\mu)$, then

1. $\lambda \in \mathcal{W}^{\mathfrak{q}}_{\text{link}} \bullet \mu$ if $\mathfrak{q}$ is of Cartan type.
2. $\lambda \in \mathcal{W}^{\mathfrak{q}}_{\text{link}} \bullet (\mu + \mathbb{Z}\Delta^{\mathfrak{q}}_{+,\text{odd}})$ if $\mathfrak{q}$ is of super type.

Idea of the proof

Lusztig isomorphisms [Heckenberger]

$w = \sigma_{i_k} \cdots \sigma_{i_1} : w^{-*}\mathfrak{q} \longrightarrow \mathfrak{q}$

$$T_w = T_{i_k} \cdots T_{i_1} : U_{w^{-*}\mathfrak{q}} \longrightarrow U_{\mathfrak{q}}$$

Rmk

$$U_{\mathfrak{q}} \simeq T_w\left(U^-_{w^{-*}\mathfrak{q}}\right) \otimes U^0_{\mathfrak{q}} \otimes T_w\left(U^+_{w^{-*}\mathfrak{q}}\right)$$

Twisted Verma modules

$$Z^w_{\mathbb{k}}(\mu) = U_{\mathfrak{q}} \otimes_{U^0_{\mathfrak{q}} T_w(U^+_{\mathfrak{q}})} \mathbb{k}^\mu$$

Theorem

Rmk

Same composition factor

$$\operatorname{ch} Z_{\Bbbk}(\mu) = \operatorname{ch} Z^{w}_{\Bbbk}(\mu\langle w\rangle)$$

$\mu + w(\rho^{w^{-1}f}) - \rho^{w^{-1}f}$

$$\operatorname{Hom}_{\mathcal{C}^{\mathfrak{q}}_{\Bbbk}}\left(Z^{x}_{\Bbbk}(\mu\langle x\rangle), Z^{w}_{\Bbbk}(\mu\langle w\rangle)\right) \simeq \Bbbk$$

Proof of the linkage principle

$L_{\Bbbk}(\lambda) < Z_{\Bbbk}(\mu) \Rightarrow L_{\Bbbk}(\lambda) < \ker \Phi$

$\operatorname{Im} \Phi \simeq L_{\Bbbk}(\mu)$

$$\begin{array}{ccccccc} & & & \xrightarrow{\Phi} & & & \\ Z_{\Bbbk}(\mu) \longrightarrow & \cdots & Z_{\Bbbk}^{w}(\mu\langle w\rangle) & \xrightarrow{\varphi} & Z_{\Bbbk}^{w\sigma_{i_s}}(\mu\langle w\sigma_{i_s}\rangle) & \cdots\longrightarrow & Z_{\Bbbk}^{w_0}(\mu\langle w_0\rangle) \\ & & \uparrow \psi & & & & \\ Z_{\Bbbk}(\beta\downarrow\mu) \longrightarrow & \cdots & Z_{\Bbbk}^{w_s}(\beta\downarrow\mu\langle w\rangle) & & & & \end{array}$$

Same Character

$\ker \varphi = \operatorname{Im} \psi$

Same character

w_0 = longest element

Outlook

- Translation functors
- 2-atypical, 3-atypical, 4-atypical, 5-atypical... simple modules
- Connection with Lie superalgebras